

Incompressible Hydrodynamics (Dana Longcope 11/16/07)

The incompressible limit of fluid mechanics is a very important and often misunderstood one. Incompressibility is an approximation which is well satisfied by flows which are very subsonic. It is therefore the appropriate limit for modeling almost all fluid dynamics with which you are familiar — even in air (especially in air). It is a system in which almost all treatments must be non-linear, so it is among the most fundamentally challenging in Physics.

We begin with the ideal fluid equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v} \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g} \quad (2)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{v} \quad (3)$$

Static equilibrium solutions to these, neglecting gravity, are a uniform pressure, p_0 . Assuming also a uniform density, ρ_0 , the linearized equations become

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}_1 \quad (4)$$

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{\nabla p_1}{\rho_0} \quad (5)$$

$$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{v}_1 \quad (6)$$

where subscript 1 designates a perturbation.

Combining $\partial/\partial t$ of the divergence of eq. (5) with eq. (6) yields a single wave equation

$$\frac{\partial^2}{\partial t^2} (\nabla \cdot \mathbf{v}) = c_s^2 \nabla^2 (\nabla \cdot \mathbf{v}) \quad , \quad (7)$$

where $c_s = \sqrt{\gamma p_0 / \rho_0}$ is the sound speed. This equation demonstrates that any divergence in the flow field is part of a sound wave, and will propagate away at the speed of sound, c_s . It is possible to consider the flow *without* any sound waves by taking $\nabla \cdot \mathbf{v} = 0$ — by studying incompressible flows.

A more rigorous way of approaching very subsonic flow is to consider the formal limit $c_s \rightarrow \infty$. That is to say, if a flow is truly slow compared to c_s , then the sound speed may as well be arbitrarily large. Applying this limit to eq. (7), without permitting infinite frequencies requires

$$\nabla^2 (\nabla \cdot \mathbf{v}) = 0 \quad . \quad (8)$$

To solve this Laplace's equation we need boundary conditions on $\nabla \cdot \mathbf{v}$. If our fluid is confined inside expanding or contracting boundaries, then $\nabla \cdot \mathbf{v}$ would be a positive or negative constant. On the other hand, in an unbounded domain, such as the outside air, the only possible solution is

$$\nabla \cdot \mathbf{v} = 0 \quad . \quad (9)$$

One tends to think of air as compressible based on experiments with pistons. But you would find it extremely difficult to compress any air by simply pushing it with your open hand. Indeed, your only hope would be to move your hand at close to the speed of sound!

Equation (9) is the condition of **incompressibility**. As we have just shown, it does not describe a kind of fluid, rather it describes the state of the flow: *Flows which are very subsonic will always be incompressible*. The smaller is the Mach number, $M \equiv v/c_s$, the better will be the approximation of incompressibility.

Once we adopt the hypothesis that \mathbf{v} has no divergence (it is incompressible) we can address the evolution of its curl, called the *vorticity*: $\boldsymbol{\zeta} \equiv \nabla \times \mathbf{v}$. Taking the curl of eq. (2) gives the simple, but disappointing equation

$$\frac{\partial(\nabla \times \mathbf{v}_1)}{\partial t} = \frac{\partial \boldsymbol{\zeta}_1}{\partial t} = 0 . \quad (10)$$

This recovers the fact that shear flow (flow with curl but no divergence) has no dynamics at linear order: they are zero-frequency modes. In order to learn more about these flows — low Mach number flows — we must study them non-linearly.

2 Equations of Incompressible Hydrodynamics

The starting point for incompressible hydrodynamics is the assumption that $\nabla \cdot \mathbf{v} = 0$. Using this in eq. (1) reveals that $D\rho/Dt = 0$: density is advected by the flow. It is possible to use complicated density functions, but the simplest, and most common version of the equations takes a uniform initial $\rho(\mathbf{x}, 0) = \rho_0$. Since this is advected, the density remains forever uniform. After that the equations of incompressible hydrodynamics are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho_0} + \mathbf{g} \quad (11)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (12)$$

We show below that these are a complete, closed, self-consistent set of equations for $\mathbf{v}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$.

2.1 The role of pressure in incompressible hydrodynamics

The most puzzling aspect of the conversion of compressible equations (1)–(3) to incompressible equations (11)–(12) is how the role of pressure changes. What happened to the equation of state, (3) and how can we find pressure without it? The simplest answer, to which we return below, is that we don't *need* to know the pressure. In spite of this we can still try to find out what it is. Taking the divergence of equation (11) now gives

$$\nabla^2 p = -\nabla \cdot [\rho_0(\mathbf{v} \cdot \nabla) \mathbf{v}] = -S(\mathbf{x}, t) , \quad (13)$$

after using (12) and then fact that $\nabla \cdot \mathbf{g} = 0$ unless the fluid is creating its own gravitational field (an interesting situation, but one we will not consider here). The expression on the right looks daunting enough, but it is nothing but a complicated function of space once we know $\mathbf{v}(\mathbf{x})$. Thus eq. (13) is a Poisson equation for pressure. In an unbounded domain we can solve it using the Green's function for ∇^2 we know well from electrostatics:

$$p(\mathbf{x}, t) = p_0 - \rho_0 g z + \underbrace{\frac{1}{4\pi} \int \frac{S(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3 x'}_{\tilde{p}(\mathbf{x}, t)} . \quad (14)$$

The second term will cancel out the gravitational field, $\mathbf{g} = -g\hat{\mathbf{z}}$, while the third, \tilde{p} , depends on the present flow field.

The pressure at a point \mathbf{x} depends on the flow *everywhere in space* at that same instant. This should seem puzzling. It means that as I drive my car, the air passing across my windshield is affecting pressure way far ahead of me — instantaneously! This action-at-a-distance is a consequence of taking the limit $c_s \rightarrow \infty$. In actual fact, the pressure out ahead of me responds only after sound waves have reached it from my windshield. Provided I drive well below the sound speed (700 m.p.h.) this time delay is very short and the effect may as well be instantaneous.

So why does the pressure depend on velocity rather than on thermodynamics (i.e. through eq. [3])? One version of the answer is that equation (12) is a *constraint* on the flow field, and p now plays the role of Lagrange multiplier, enforcing that constraint. This is analogous to a pendulum bob suspended by an inextensible string of length ℓ . The string tension must vary as the pendulum swings. In an extensible string that tension force is proportional to the extension of the string by Hooke's law: $F_r = -k(r - \ell)$. This extensibility leads to a kind of very fast normal mode, $m\ddot{r} = -k(r - \ell)$ whose frequency of oscillation is $\sqrt{k/m}$. Assuming the string to be inextensible means replacing Hooke's law with a constraint, $r = \ell$, and a constraint force complete with Lagrange multiplier. This is a good approach when the period of the pendulum is *much, much* longer than the Hooke's law oscillations: $k/m \gg g/\ell$. In direct analogy, we use incompressible hydrodynamics when the motion of the flow is *much, much* slower than the motion of sound waves. We thus replace the compression modes, contained in eq. (3), with constraint equation (12).

It is also instructive to consider the typical magnitude of the pressure contribution, \tilde{p} . The source term in (13) has typical magnitude $S \sim \rho_0 v^2 / L^2$ where v is the typical flow speed and L is a typical length scale. Using this in the integral from (14) shows that

$$\frac{\tilde{p}}{p_0} \sim \frac{\rho_0 v^2}{p_0} \sim \frac{v^2}{c_s^2} = M^2 \ll 1 . \quad (15)$$

The flow therefore leads to fluctuations in the pressure field which are smaller than the ambient pressure p_0 , by $M^2 \ll 1$. If it is so small why is it not dropped from eq. (11)? The answer is simply that the other two terms in (14) disappear from the right hand side, leaving only $-\nabla\tilde{p}/\rho_0$. This is small, in some sense, but it is exactly as small as the terms of the left hand side.

How can we reconcile these fluctuation in pressure with incompressibility: $\nabla \cdot \mathbf{v} = 0$? Equation (3) says that

$$\frac{D}{Dt} \left(\frac{p}{p_0} \right) = \frac{D}{Dt} \left(\frac{\tilde{p}}{p_0} \right) - \frac{\rho_0 g}{p_0} v_z \simeq -\gamma \nabla \cdot \mathbf{v} ,$$

which should vanish since $\nabla \cdot \mathbf{v} = 0$. What incompressibility really means is that $\nabla \cdot \mathbf{v}$ is very, very small and can therefore be approximated as zero. This is consistent with the terms on the left of these relations, which are both very small provided or system size is $L \ll H_p = p_0 / \rho_0 g$ the scale height of the atmosphere.

The fluid must therefore undergo a very small amount of compression, $\sim M^2$, in order for the pressure to keep the flow incompressible. If this sounds contradictory, recall the inextensible string whose length was fixed at ℓ . In actual fact the string undergoes a very small extension in order for Hooke's law, $F_r = -k(r - \ell)$, to produce the required force (a.k.a. force of constraint). Provided the string is very stiff, $k \rightarrow \infty$, we may neglect this by saying $r - \ell \simeq 0$.

Won't that same compression also change the density, contradicting our assumption that $\rho = \rho_0$? Yes, but note that $\nabla\rho$ does not appear in any of the fluid equations. We can thus expand $\rho = \rho_0 + \delta\rho$, and drop the perturbation everywhere since $\delta\rho \sim M^2\rho_0$.

2.2 The vorticity equation

The most informative way to study incompressible fluid dynamics is to eliminate the pressure from them at the outset. Since $p(\mathbf{x}, t)$ is a kind of Lagrange multiplier, enforcing a constraint, we may wish to use coordinates where the constraint is unnecessary. We first use vector identities to rewrite

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) .$$

With this we can rewrite the momentum equation, (11), in the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho_0} + gz \right) . \quad (16)$$

Taking the curl of this entire equation will eliminate the term involving pressure as well as gravity. Using the vorticity $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$, we can write the resulting equation, called the *vorticity equation*,

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\zeta}) = 0 \quad (17)$$

$$\nabla \times \mathbf{v} = \boldsymbol{\zeta} , \quad (18)$$

where the second is just a restatement of the relation between \mathbf{v} and $\boldsymbol{\zeta}$. The two equations form a complete, coupled system for variables \mathbf{v} and $\boldsymbol{\zeta}$, of which pressure is not a part. Thanks to the second term in (17) they are non-linear, and will therefore be impervious to most known solution methods.

Equation (17) can be written in a slightly more intuitive way using one more vector identity:

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\zeta} = \frac{D\boldsymbol{\zeta}}{Dt} = (\boldsymbol{\zeta} \cdot \nabla)\mathbf{v} . \quad (19)$$

This shows that vorticity can be changed by the flow itself. If the matrix $\partial v_i / \partial x_j$ has a positive eigenvalue, then $\boldsymbol{\zeta}$ will be amplified in the direction of that eigenvector. This is known as *vortex stretching*. Tornadoes are a well-known manifestations of very high concentrations of vorticity. They are created by a vortex stretching process. The more common (and less dangerous) phenomenon of water swirling as it leaves a drain is another instance of vortex stretching. In that case $\partial v_z / \partial z \equiv \omega > 0$ since the water speeds up descending the drain pipe. The $\hat{\mathbf{z}}$ component of equation (19), $D\zeta_z / Dt = \omega \zeta_z$, shows that the observed swirling, ζ_z , should speed up with time.

3 Two-dimensional, incompressible fluid dynamics

The most illuminating setting for hydrodynamics is to consider flows within the x - y plane. That is to say

$$\mathbf{v} = v_x(x, y, t)\hat{\mathbf{x}} + v_y(x, y, t)\hat{\mathbf{y}} . \quad (20)$$

The condition of incompressibility, $\nabla \cdot \mathbf{v} = \partial v_x / \partial x + \partial v_y / \partial y$ can be satisfied by defining the velocity in terms of a new function, $\psi(x, y, t)$ known as the *stream function*. The velocity is defined in terms of derivatives of this function

$$\mathbf{v} = \nabla\psi \times \hat{\mathbf{z}} = \frac{\partial \psi}{\partial y}\hat{\mathbf{x}} - \frac{\partial \psi}{\partial x}\hat{\mathbf{y}} , \quad (21)$$

so that $\nabla \cdot \mathbf{v} = 0$ for any function $\psi(x, y, t)$.

The stream function ψ must never be confused with the velocity potential Φ , whose gradient is used to generate *irrotational* flow: $\mathbf{v} = -\nabla\Phi$. Both are scalar functions whose derivatives give velocity components, but the parallels end there. In fact the two scalar functions serve completely opposite purposes. Irrotational flow, $\mathbf{v} = -\nabla\Phi$, is composed of *nothing but* sound waves, while incompressible flow, $\mathbf{v} = \nabla\psi \times \hat{\mathbf{z}}$, includes *no sound waves whatsoever*. They are two mutually exclusive situations. The most general two-dimensional flow would include both components: $\mathbf{v} = \nabla\psi \times \hat{\mathbf{z}} - \nabla\Phi$. This representation is known as a Helmholtz decomposition; it is a representation of a vector field as a sum of a divergence-free component and a curl-free component.

Replacing two unknown functions, v_x and v_y , with a single stream function, ψ , will cut our work in half. More important still, the stream function allows us to understand the flow at a glance. An ancillary result of (21) is that

$$\mathbf{v} \cdot \nabla\psi = (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla\psi = 0 .$$

This means that *velocity vectors are everywhere parallel to the contours of the stream function*. These contours are called *streamlines*, and they show at a glance how the fluid is flowing. Where streamlines are close together, $|\nabla\psi|$ is large and the flow is fast. Closed curves around local maxima (minima) show paths of circular flow, called eddies, flowing clockwise (counter-clockwise).

To complete the system we take the curl of (21) to find the vorticity

$$\boldsymbol{\zeta} = \nabla \times \mathbf{v} = -\nabla^2\psi \hat{\mathbf{z}} = \zeta \hat{\mathbf{z}} . \quad (22)$$

The vorticity is therefore a single scalar function, $\zeta(x, y, t)$, whose evolution is determined by the vorticity equation

$$\frac{\partial\zeta}{\partial t} + (\mathbf{v} \cdot \nabla)\zeta = \frac{D\zeta}{Dt} = 0 , \quad (23)$$

from eq. (19). Thus the function $\zeta(x, y, t)$ is carried around with the fluid.

3.1 Steady two-dimensional flows

For a flow to be equilibrium, or steady, its vorticity must remain unchanging $\partial\zeta/\partial t = 0$. Using this in (23) gives

$$(\mathbf{v} \cdot \nabla)\zeta = (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla\zeta = \frac{\partial\zeta}{\partial x} \frac{\partial\psi}{\partial y} - \frac{\partial\zeta}{\partial y} \frac{\partial\psi}{\partial x} = 0 . \quad (24)$$

In order to satisfy this equation we seek a function $\psi(x, y)$ whose Laplacian has the same functional form that it does:

$$\nabla^2\psi = F(\psi) , \quad (25)$$

where $F(u)$ is any function of a single variable. Equation (25) leads to a gradient

$$\nabla\zeta = -F'(\psi) \nabla\psi ,$$

which will clearly satisfy eq. (24), no matter what the function $F'(\psi)$ is. Whether or not you will be able to solve eq. (25) depends entirely on how you choose the function $F(\psi)$. Particularly simple choices are $F(u) = 0$ or $F(u) = -\lambda u$ for some constant λ .

One particular example of a complicated flow is that generated by the stream function

$$\psi(x, y) = \ln[\cosh(y) - \xi \cos(x)] \quad , \quad (26)$$

where ξ is a parameter ($0 \leq \xi \leq 1$). Contours of this function, shown in Fig. 1, show an upper layer of rightward flow and a lower layer of leftward flow sandwiching a chain of clockwise eddies. It is worth taking time to understand how expression (26) leads to Fig. 1 — there is a very similar pattern in Hamiltonian mechanics. Where are the centers of the eddies? Where are the stagnation points between them? A truly remarkable feature of expression (26) is that taking derivatives gives

$$\nabla^2 \psi = \frac{1 - \xi^2}{[\cosh(y) - \xi \cos(x)]^2} = -(1 - \xi^2)e^{-2\psi} \quad .$$

The flow therefore, satisfies eq. (25) for the choice $F(u) = (\xi^2 - 1)e^{-2u}$, so it is a steady flow. Apparently the eddies will remain in place, between opposing layers of horizontal flow.

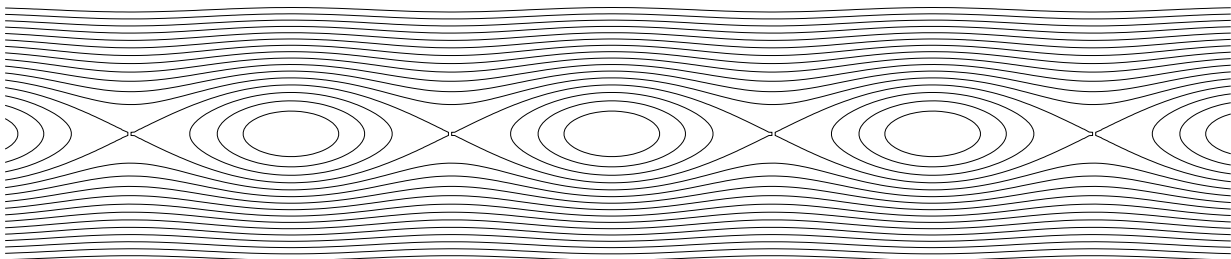


Figure 1. Streamlines for the flow given by eq. (26), with $\xi = 0.25$.

3.2 Steady flow over an object

Let us now specialize one final time to consider the flow of a fluid, such as air, around a solid object like a car or an airfoil. This is an equivalent problem to the flow around that object as it moves through still air, but viewed from the reference frame of the object. If the motion is subsonic we may use incompressible equations; if it is moving perpendicular to its own long axis we may consider that flow to be two-dimensional. We assume the flow far from the object is uniform: $\mathbf{v} = \mathbf{V}_0$ (see Fig. 2) which is consistent with a co-moving view of the object itself moving at velocity $-\mathbf{V}_0$.

Uniform flow means that $\nabla \times \mathbf{v} = \zeta \hat{\mathbf{z}} = 0$ far upstream. In two dimensional flow the vorticity, $\zeta = 0$, will be preserved as it is brought near the object. This means that the entire flow is described by a harmonic streamfunction

$$\nabla^2 \psi = -\zeta = 0 \quad . \quad (27)$$

This is a special case of (25) for $F(u) = 0$, so the flow will be steady (an equilibrium).

Equation (27) is Laplace's equation, whose solutions should be well known to you. The surface of the object itself is a closed curve \mathcal{C} , and eq. (27) applies everywhere outside it. Placing the origin inside \mathcal{C} , the solution to Laplace's equation can be expressed in cylindrical coordinates

$$\psi(r, \phi) = V_0 r \sin(\phi - \alpha) + A_0 \ln(r) + B_0 + \sum_{m=1}^{\infty} r^{-m} [A_m \cos(m\phi) + B_m \sin(m\phi)] \quad , \quad (28)$$

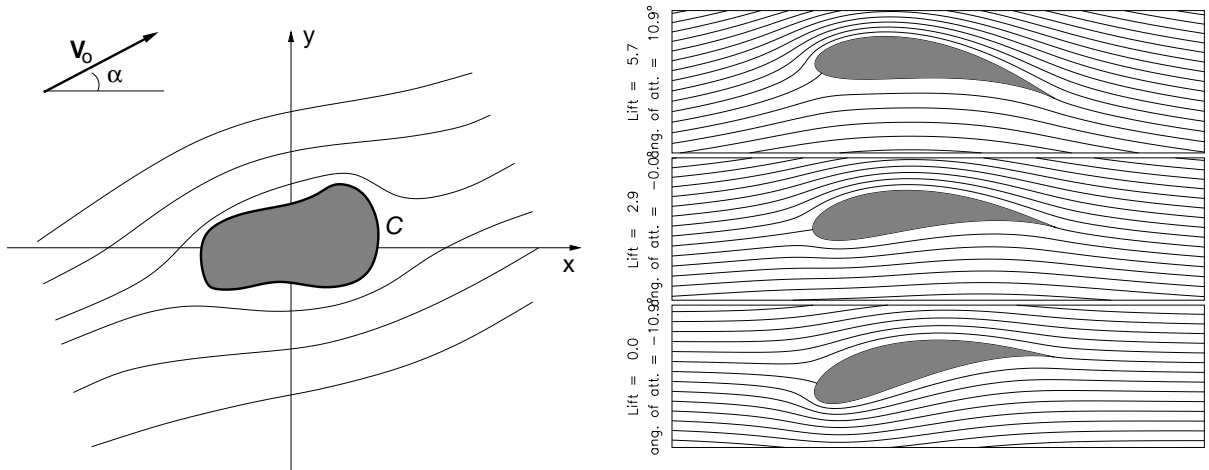


Figure 2. Steady flow over an object (grey) surrounded by a closed curve \mathcal{C} (dark line). Curves show the streamlines of the flow, which are contours of $\psi(r, \phi)$. Upstream (to the left) the flow is uniform at an angle α . (left) The generic case where upstream flow direction is shown by an arrow. (right) Three different cases of horizontal flow ($\alpha = 0$) across an airfoil (grey) at different orientations.

where the constants A_m and B_m are fixed by the boundary conditions. There is no flow inside the object so none of the divergences from $\ln(r)$ or r^{-m} will appear in our solution. At large distances, $r \rightarrow \infty$, the solution will be dominated by the first term in (28)

$$\psi \simeq V_0 y \cos(\alpha) - V_0 x \sin(\alpha) ,$$

from which $\nabla\psi \times \hat{\mathbf{z}} = V_0(\hat{\mathbf{x}} \cos \alpha + \hat{\mathbf{y}} \sin \alpha)$. This is the uniform upstream flow, directed at an angle α to the x axis.

The remainder of the terms in (28) describe the way the fluid flows around the object. The flow must be everywhere tangent to \mathcal{C} . Since the flow is everywhere tangent to contours of $\psi(x, y)$, the curve \mathcal{C} must be a contour — we can take it to be the $\psi = 0$ contour. In other words Laplace's equation, (27), is subject to the Dirichlet boundary condition

$$\psi = 0 \quad , \quad \mathbf{x} \in \mathcal{C} . \quad (29)$$

This condition is used to determine the coefficients, A_m and B_m in eq. (28).

A particularly simple object is a cylinder of radius a . Satisfying (29) on the curve, $r = a$, leads to the solution

$$\psi(r, \phi) = V_0 \left[r - \frac{a^2}{r} \right] \sin(\phi - \alpha) + A_0 \ln(r/a) , \quad (30)$$

whose streamlines are shown in Fig. 3. The first term on the left (see left of Fig. 3) may be familiar from electrostatics: it is the electrostatic potential when a conducting cylinder is placed in a uniform electric field. In the second term, the constant B_0 was set to $B_0 = -A_0 \ln a$, but one constant, A_0 , remains free. It is easy to verify that $\psi(a, \phi) = 0$ regardless of what value A_0 has. Moreover, far from the cylinder, $r \gg a$, the flow field becomes

$$\mathbf{v} \simeq V_0(\hat{\mathbf{x}} \cos \alpha + \hat{\mathbf{y}} \sin \alpha) - \frac{A_0}{r} \hat{\phi} , \quad (31)$$

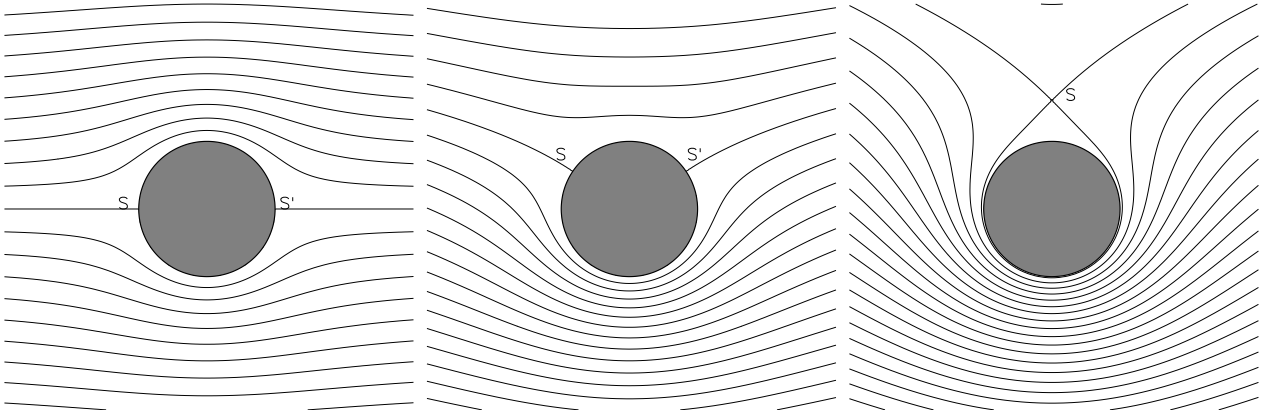


Figure 3. Stream lines of horizontal flow, $\alpha = 0$, around a cylinder. Each panel shows contours of $\psi(r, \phi)$ from eq. (30), for different values of $A_0 = -\Gamma/2\pi$. From left to right they have $\Gamma = 0$, $\Gamma = 7V_0a$, and $\Gamma = 14V_0a$. In each case there are stagnation points, labeled S or S' where $\mathbf{v} = 0$.

which matches the asymptotic flow regardless of A_0 .

So what is the physical meaning of A_0 in the flow described by (28) or (30)? The circulation around a closed path, \mathcal{P} , is defined as

$$\Gamma \equiv \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l} = \oint_{\mathcal{P}} \nabla\psi \cdot (\hat{\mathbf{z}} \times d\mathbf{l}) \quad . \quad (32)$$

If $\Gamma > 0$ there is a counter-clockwise flow around the path \mathcal{P} . According to Kelvin's theorem Γ is conserved as the path \mathcal{P} is advected with the fluid.

Since $\nabla^2\psi = 0$ everywhere outside the object, it is possible to show that every possible path around it will have the same circulation. Performing the integral for (28) gives $\Gamma = -2\pi A_0$ for any path enclosing the object once. This integral is particularly easy to perform at very large radii where expression (31) obtains. The flows in the center and right of Fig. 3 have $\Gamma > 0$, meaning that there is a counter-clockwise circulation. This is due to the faster flow along the bottom of the cylinder, where the contours are closer.

So while the flow has no vorticity, $\zeta = -\nabla^2\psi = 0$, it is not irrotational: it has a circulation unless $A_0 = 0$. The different values of A_0 describe flow around the object with different circulation. Each is a valid flow around our object from uniform upstream flow.

3.3 Lift: the Magnus force

One of the consequences of flow with circulation is a force on the object at right angles to the flow. From the version of the momentum equation, eq. (16), for steady flow ($\partial\mathbf{v}/\partial t = 0$) we can readily show

$$\mathbf{v} \cdot \nabla \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho_0} + gz \right) = 0 \quad . \quad (33)$$

This is Bernoulli's principle for incompressible flow, stating that the expression in parentheses will be constant along each streamline. Since all streamlines originate far upstream, where the flow is

uniform, the value will be set there. Ignoring gravity for the moment¹ this gives the result that

$$p(x, y) = p_0 + \frac{1}{2}\rho_0 V_0^2 - \frac{1}{2}\rho_0 |\mathbf{v}|^2 \quad (34)$$

where p_0 is the ambient pressure far upstream. Where the flow is fast the pressure will be low. The low pressure can be viewed as the cause of the fast flow rather than its effect: the fluid sped up as it fell down the pressure gradient. Similarly the pressure will be highest at a stagnation point, $\mathbf{v} = 0$; the high pressure was responsible for stopping the fluid.

The pressure distributed over the object's surface, \mathcal{C} , will exert a force on it. Since we are working in two dimensions, this will actually be a force per unit length; the total force is found by multiplying by the object's length in the z direction. We begin by calculating the force per unit length exerted on the surface \mathcal{C}

$$\mathbf{F} = \oint_{\mathcal{C}} \hat{\mathbf{n}} p(x, y) dl \quad , \quad (35)$$

where $\hat{\mathbf{n}}$ is the normal directed outward from the body; the normal on which the pressure is acting is $-\hat{\mathbf{n}}$, since the fluid is *outside* the curve \mathcal{C} . The first two terms in eq. (34) will not contribute to this integral since they are uniform. This leaves the net force

$$\mathbf{F} = -\frac{1}{2}\rho_0 \oint_{\mathcal{C}} \hat{\mathbf{n}} |\mathbf{v}|^2 dl \quad , \quad (36)$$

depending only on the speed of the fluid at the object's surface.

In order to use expression (36) we would need to know the complete solution, $\psi(x, y)$, at least in the neighborhood of the object. Let us use the case of the cylinder as an illustration, since (30) gives the complete solution. Taking the derivative and *then* evaluating at $r = a$ we find

$$\mathbf{v}(a, \phi) = -2V_0 \sin(\phi - \alpha) \hat{\boldsymbol{\phi}} - \frac{A_0}{a} \hat{\boldsymbol{\phi}} \quad . \quad (37)$$

Since the flow must be tangent to the surface, there is no radial component. It is also noteworthy that for cases with $|A_0| < 2V_0 a$ (i.e. $|\Gamma| < 4\pi V_0 a$) there are two stagnation points on the surface; as there are in the center and left panels of Fig. 3. Using expression (37) in the force, eq. (36) gives

$$\begin{aligned} \mathbf{F} &= -\frac{1}{2}\rho_0 \int_0^{2\pi} \hat{\mathbf{r}} [2V_0 \sin(\phi - \alpha) + A_0/a]^2 a d\phi \\ &= -\frac{1}{2}\rho_0 \int_0^{2\pi} [\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}] 4V_0 A_0 \sin(\phi - \alpha) d\phi \\ &= -2\pi \rho_0 V_0 A_0 [\cos \alpha \hat{\mathbf{y}} - \sin \alpha \hat{\mathbf{x}}] \quad , \end{aligned} \quad (38)$$

since terms proportional to V_0^2 and A_0^2 both integrate to zero. We can use the relation $A_0 = -\Gamma/2\pi$ to write this in a simple physical form

$$\mathbf{F} = \rho_0 \Gamma \mathbf{V}_0 \times \hat{\mathbf{z}} \quad , \quad (39)$$

known as the Kutta-Joukowski formula.

¹It is easy to show that the integral of $\hat{\mathbf{n}}\rho_0 g z$ around the surface of the object yields the buoyancy force. Integration by parts converts the path integral to an integral of $\rho_0 g \hat{\mathbf{z}}$ over the area inside \mathcal{C} .

This is the force on the surface of the cylinder due to the pressure distributed about it. For the central and right panels in Fig. 3, where $\Gamma > 0$ and $\mathbf{V}_0 = V_0 \hat{\mathbf{x}}$, there is a downward force ($\propto -\hat{\mathbf{y}}$) on the cylinder. This is due to the higher pressure along the top where the stagnation points (labeled S and S') are. The contours are closest together along the bottom, indicating that the flow is fastest and the pressure is lowest.

We derived expression (39), known as the Magnus force, for the special case of a cylinder, eq. (30). It turns out to apply to the general solution, eq. (28), about an object of arbitrary shape. It is rather difficult to show this using (36) when \mathcal{C} is very complicated, such as an airfoil (see Fig. 2). It can, however, be demonstrated using the momentum of the upstream and downstream flow.

Let us consider the special case of purely horizontal flow ($\alpha = 0$) incident on an arbitrary object, \mathcal{C} . The flow far from the object has the same form, eq. (31),

$$\mathbf{v} \simeq V_0 \hat{\mathbf{x}} + \frac{A_0 y}{r^2} \hat{\mathbf{x}} - \frac{A_0 x}{r^2} \hat{\mathbf{y}} , \quad (40)$$

independent of the object's shape. A column of fluid Δx wide, extending over all y , will have a vertical momentum²

$$P_y = \Delta x \int_{-\infty}^{\infty} \rho_0 v_y dy = -\Delta x \rho_0 A_0 x \int_{-\infty}^{\infty} \frac{dy}{x^2 + y^2} = -\pi \Delta x \rho_0 A_0 \frac{x}{|x|} . \quad (41)$$

Remarkably, every column upstream of the object, $x/|x| = -1$, has exactly the same vertical momentum. Every column downstream, $x/|x| = +1$ has the same momentum, but it is different from the upstream momentum by

$$\Delta P_y = -2\pi \Delta x \rho_0 A_0 = \rho_0 \Gamma \Delta x . \quad (42)$$

Since these columns are moving horizontally at speed V_0 , their momenta are being changed at a rate

$$\frac{dP_y}{dt} = \rho_0 \Gamma V_0 . \quad (43)$$

This change in momentum requires a vertical force on the fluid — a force from the object. According to Newton's third law there must be an equal and opposite force, $F_y = -\rho \Gamma V_0$, on the object from the fluid. This is the Magnus force as given by (39), but here for the case $\mathbf{V}_0 = V_0 \hat{\mathbf{x}}$.

The first fact to notice about the Magnus force, eq. (39), is that it is *perpendicular* to the upstream flow direction, \mathbf{V}_0 . *It is not a drag force* (i.e. “air resistance”) which would be directed parallel to \mathbf{V}_0 . One example of the Magnus force is the *lift force* which keeps airplanes aloft. Horizontal flow over the wing creates an upward force. Another is the force on a spinning baseball which can make it curve unexpectedly on its way to home plate: it makes a curve ball curve.

The general derivation above gives some physical insight into the origin of the force. There is a force on the object because it is *deflecting the airflow*. More simply put the object is *pushing* against the fluid and the fluid is pushing back. It is pushing at right angles to the flow, not against it, and therefore it deflects the flow but does not slow it down. The amount of deflection depends on the circulation, Γ , about the object, and so therefore does the Magnus force. In the upper two panels on the right of Fig. 2, $\Gamma < 0$, so there an upward lift force. The air comes in from the left moving slightly upward, and leaves to the right going slightly downward. The lift on the wing was exerted in the course of imparting this downward momentum to the air.

²This is actually a momentum per unit length in z . Its time-rate-of-change will give the force per unit length, \mathbf{F} .

3.4 Whither the drag force?

The absence of a drag force from steady flow without vorticity is known as d’Alembert’s paradox. So what happened to the drag force? You might suspect it was lost when we neglected the fluid’s viscosity, and you would not be entirely wrong. But the viscosity of air is extremely small, and it’s hard to believe such a small effect can explain why big cars use so much gasoline. The main ingredient needed to recover a drag force is not viscosity, but rather some kind of *wake* in the flow field: a column of fluid behind the body whose streamlines *do not* originate upstream — in extreme cases the fluid is *stationary* in the frame of object. Since its streamlines do not originate upstream, the pressure inside the wake cannot be found from Bernoulli’s principle; it is typically $p \sim p_0$ throughout when $|\mathbf{v}| = 0$. The pressure at a leading stagnation point will be greater than this, $p \sim p_0 + \frac{1}{2}\rho_0 V_0^2$, giving a net backward force on the object. This is the drag force.

For a flow to include a wake its stream function, ψ , should take on a roughly constant value within a range of angles around $\phi \sim \alpha$. This is not the case for our general solution to Laplace’s equation (28), since $\psi \sim V_0 r \sin(\phi - \alpha)$. We might have noted earlier that in the general solution (28), the flow is the same far from the object, $\mathbf{v} \rightarrow \mathbf{V}_0$, whether you look upstream, *downstream* or to the side — there is no wake disturbing any part of the flow downstream. Solution (28) has no wake so it therefore has no drag. This is the origin of d’Alembert’s paradox.

A more practical consequence of d’Alembert’s paradox is that there will be no drag if the object moves through the fluid without leaving a disturbance behind it (i.e. a wake). The smaller the disturbance you create the smaller will be the drag force you face. This means that streamlining an object, to minimize its drag, is first and foremost about the fluid flow *behind* the object — its wake. Naive people, feeling the air pressure on their front, will focus streamlining efforts there: “make the front pointier so there is less drag.” Aerodynamic engineers know, however, that the secret to streamlining is in the back. Looking at a commercial airliner you will notice that the back end is far pointier than the front.

We may use our experience with harmonic functions to say more generally that any function which is uniform over some region (the wake) but not elsewhere (upstream) cannot be a solution of Laplace’s equation. Thus we see that in a flow with a wake $\nabla^2\psi = -\zeta$ cannot be zero everywhere: there must be vorticity somewhere in the flow. In particular, there must be layers of vorticity separating the stationary wake from the rest of the downstream flow. But according to eq. (23) this vorticity cannot lie on streamlines from upstream. Where then did it come from? Typically these vortex layers connect to a piece of the body itself. A very thin layer, called a *boundary layer* breaks free from the object’s surface and forms the outside of the wake. We ruled out such behavior when we took $\nabla^2\psi = 0$ everywhere in the flow. A thorough understanding of this boundary layer does require the inclusion of viscosity, so it does play a role. But what receives the momentum when the air exerts a force on the body? The wake does.

3.5 Lift on an airfoil

The foregoing section showed the Kutta-Joukowski formula, eq. (39) for the Magnus force. This provides a ready answer to the question “What makes an airplane fly?”. The answer it gives, “Circulation on its wings”, is correct, but simply raises one more question: “Why do the wings have circulation?”. This question is even more troubling than the first, since the circulation was a free parameter in our analysis. At least in solution (30) the value of A_0 can be anything and it will still satisfy the equation, (27), and the boundary condition. If this were the case for an airfoil,

then the lift could be anything — even zero.

It turns out there is a way to find a unique value of circulation for an airfoil in a given upstream flow. Doing so exactly would require solving Laplace’s equation with boundary conditions, $\psi = 0$, along a very complicated curve, \mathcal{C} , such as one on the right of Fig. 2. This is possible using very sophisticated analysis techniques (that’s how the figures were made). The basic idea, however, requires only a little knowledge about Laplace’s equation and a single feature common to all airfoils.

The problem facing us is to find a function $\psi(r, \phi)$ satisfying Laplace’s equation, eq. (27), and homogeneous boundary conditions, $\psi = 0$ on the curve \mathcal{C} . The general solution, (28) will do this for the appropriate choice of coefficients A_m and B_m , $m \geq 1$. Let us assume we have found two such solutions: ψ_α and ψ_c . The first is defined have zero circulation ($A_0 = 0$) and approach

$$\psi_\alpha(r, \phi) \rightarrow r \sin(\phi - \alpha) \quad , \quad r \rightarrow \infty \quad . \quad (44)$$

This means it has flow speed unity making an angle α . The second solution will have unit circulation, and approach

$$\psi_c(r, \phi) \rightarrow -\frac{\ln(r)}{2\pi} \quad , \quad r \rightarrow \infty \quad , \quad (45)$$

so it has no net flow far from the airfoil. Both functions are solutions to Laplace’s equation, so any superposition

$$\psi(r, \phi) = V_0 \psi_\alpha(r, \phi) + \Gamma \psi_c(r, \phi) \quad , \quad (46)$$

will be as well. This particular superposition will be a flow with lift force $-\rho_0 V_0 \Gamma$ directed upward (i.e. it will be upward provided $\Gamma < 0$).

The foregoing simply restates our previous conclusion that the circulation, Γ , is a free parameter of the solution. For a given upstream flow speed, V_0 , and angle α , we may add any multiple of ψ_c to get any circulation we want, and therefore any lift we want.

The analysis is not so simple when the curve \mathcal{C} has a discontinuity in its tangent — a sharp corner. If it has a sharp corner, as all good airfoils do, then general solutions to Laplace’s equation will have a singularity there. To be sure ψ will vanish over the entire curve, but its derivative, $|\nabla\psi|$, can diverge approaching the sharp corner, as it does in Fig. 4. This same kind of singularity occurs in electrostatic fields near a pointed conductor. Large electric field there can be the source of field emission. In this case the divergence means that $|\mathbf{v}| \rightarrow \infty$ around the sharp corner — a troubling property indeed.

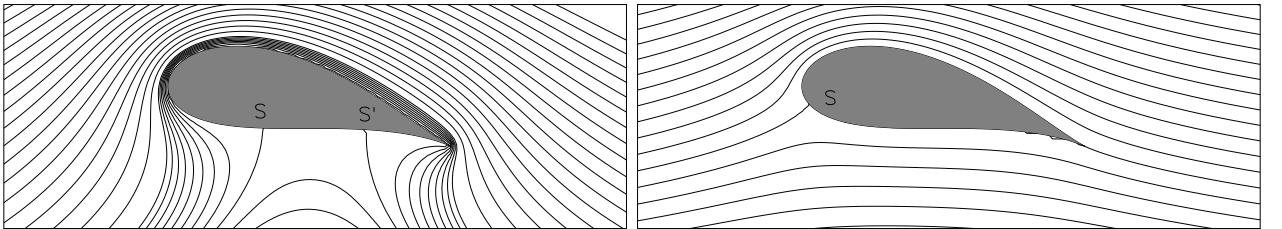


Figure 4. Two different airflows with the same upstream velocity but different circulations, Γ . (left) A flow which is singular at the trailing edge. A higher density of contours are plotted near $\psi = 0$ in order to better visualize the flow in the vicinity of the airfoil’s surface. Evidence that $\nabla\psi \rightarrow \infty$ at the trailing edge is found in the very high density of contours there. (right) The flow, ψ_{reg} which is regular at the trailing edge. Note that the second stagnation point, S' , present in the singular flow is absent from the regular flow.

In general there will be singularities of this same form in both solutions, ψ_α and ψ_c . What is remarkable is that it is possible to make a linear combination of the two solutions,

$$\psi_{\text{reg}}(r, \phi) = \psi_\alpha(r, \phi) - \ell_\alpha \psi_c(r, \phi) , \quad (47)$$

in which the two singularities cancel one another. There is a unique coefficient, ℓ_α (whose units happen to be length) for which this cancelation occurs, and the resulting function, $\psi_{\text{reg}}(r, \phi)$, has a bounded gradient, even at the sharp corner (see right panel of Fig. 4).

Every linear combination of the form (46) satisfies the equations of incompressible hydrodynamics — even the singular ones. Both plots in Fig. 4 are valid solutions of Laplace’s equation, satisfying all the boundary conditions. The particular combination, $\psi_{\text{reg}}(r, \phi)$, however, is the only one whose velocity is finite everywhere (this is the right panel). Joukowski proposed that this non-singular flow would be the only one actually observed. It seems like a reasonable hypothesis since arbitrarily large velocity, in Bernoulli’s principle eq. (34), would demand negative gas pressure — not a likely event. On the other hand, no real object has corners as sharp as a mathematical discontinuity. Trailing edges on real airfoils curve smoothly (they are made of metal, after all), but over radii much smaller than the rest of the wing. Joukowski’s hypothesis, even in this case, is that the actual flow will be the one for which the velocity is relatively small at the corner.

There is a unique streamfunction which matches our upstream velocity condition *and* satisfies Joukowski’s hypothesis:

$$\psi(r, \phi) = V_0 \psi_{\text{reg}} = V_0 \psi_\alpha(r, \phi) - V_0 \ell_\alpha \psi_c(r, \phi) . \quad (48)$$

Comparing this to expression (46) shows that the flow has circulation, $\Gamma = -V_0 \ell_\alpha$. In other words, under Joukowski’s hypothesis the circulation is not actually a free parameter, it is fixed by the shape of the airfoil and the upstream flow. Using this in the Kutta-Joukowski formula gives the upward lift force (per unit length) on the airfoil

$$F_{\text{lift}} = \rho_0 \ell_\alpha V_0^2 .$$

The coefficient ℓ_α comes from actually solving Laplace’s equation over the airfoil, which we have vowed not to do. It seems likely that its value will depend on the angle of the upstream flow, i.e. on α . It is also likely that it might be negative, although that would be undesirable if you plan to keep your plane in the air.

To gain one more bit of insight into the lift force, consider further decomposing the function ψ_α as

$$\psi_\alpha(r, \phi) = \cos \alpha \psi_x(r, \phi) + \sin \alpha \psi_y(r, \phi) . \quad (49)$$

ψ_x and ψ_y are just version of ψ_α for $\alpha = 0$ and $\pi/2$ respectively. Following the logic above we find two coefficients, ℓ_x and ℓ_y , such that the linear combinations

$$\psi_x(r, \phi) - \ell_x \psi_c(r, \phi) , \quad \psi_y(r, \phi) - \ell_y \psi_c(r, \phi) , \quad (50)$$

are each regular at the trailing edge. We may then superpose these two regular functions to give a regular function with an arbitrary angle to its upstream flow:

$$\psi(r, \phi) = V_0 \psi_\alpha(r, \phi) - V_0 [\cos \alpha \ell_x + \sin \alpha \ell_y] \psi_c(r, \phi) . \quad (51)$$

The coefficient of ψ_c may once again be equated to the circulation. Defining the combinations

$$\ell_0 \equiv \sqrt{\ell_x^2 + \ell_y^2} \quad , \quad \alpha_0 \equiv -\tan^{-1}(\ell_x/\ell_y) \quad , \quad (52)$$

allows the lift force to be written

$$F_{\text{lift}} = \rho_0 \ell_0 V_0^2 \sin(\alpha - \alpha_0) \quad . \quad (53)$$

The parameters α_0 and ℓ_0 depend on the shape of the specific airfoil, and require the solution of Laplace's equation.

Expression (53) reveals several universal facts about the lift force on an airfoil, even without knowing the precise values of the parameters. There is a particular angle, α_0 , from which flow will produce no lift at all (see the lower right panel of Fig. 2). As long as the flow is incident from below this direction, $\alpha > \alpha_0$, there will be an upward lift force (see the other two panels). For small values, the lift will be proportional to this difference in angles. The other parameter ℓ_0 , depends on the shape and size of the airfoil. It has dimensions of length, so it will be proportional to a linear dimension of the airfoil's cross section.

An actual wing will extend a finite length L in the third dimension (z). The total force on this wing is found by multiplying the lift-per-length, from eq. (53) by L

$$\text{Lift} = \rho_0 \ell_0 L V_0^2 \sin(\alpha - \alpha_0) \quad . \quad (54)$$

The product $\ell_0 L$ is proportional to the total area of the wing. A larger wing creates more lift. Flying faster also creates more lift. Except for the dependence on angle we might have predicted exactly this relationship from dimensional analysis alone. Indeed, the drag force on the wing, which we cannot find from our vorticity-free treatment, will be proportional to exactly the same factor. It is a force, so it must involve exactly the same combination of factors.

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