Notes on Magnetohydrodynamics

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The equations and what they describe

Magnetohydrodynamics (MHD) describes the "slow" evolution of an electrically conducting fluid — most often a *plasma* consisting of electrons and protons (perhaps seasoned sparingly with other positive ions). In MHD "slow" means evolution on time scales longer than those on which individual particles are important, or on which the electrons and ions might evolve independently of one another. The formal frequency range in which we must work is

$$\text{slow} \iff \frac{\partial}{\partial t} \ll \begin{cases} \omega_{pe} \equiv \sqrt{\dfrac{4\pi e^2 n_e}{m_e}} & \text{plasma frequency} \\ \Omega_i \equiv \dfrac{eB}{cm_p} & \text{Ion gyro-frequency} \\ \nu_{ei} \equiv \dfrac{\pi n_e e^4 \ln\Lambda}{\sqrt{m_e}(k_{\rm B}T)^{3/2}} & \text{collision frequency} \end{cases}$$

				frequencies rad/sec			
	$n_e\,[\text{cm}^{-3}]$	$T\,[\text{K}]$	$B\,[\text{G}]$	ω_{pe}	Ω_i	ν_{ei}	MHD
magnetotail	1	10^7	10^{-4}	10^5	0.1	10^{-9}	$\Delta t \gg 10$ sec†
Solar corona (AR)	10^9	10^6	10^2	10^9	10^6	10^2	$\Delta t \gg 10$ ms
Solar interior (200 Mm)	10^{23}	10^6	10^5	10^{16}	10^9	10^{16}	$\Delta t \gg 10$ ns
Galaxy	1	10^4	10^{-6}	10^5	10^{-2}	10^{-6}	$\Delta t \gg 10$ days

† Neglecting collisions

Fundamental fields Owing to their slow evolution it is possible to consider the combined electrons and ions as a single fluid. The instantaneous state of this fluid is characterized by the following fields

$\rho(\mathbf{x}, t)$ Mass density. On these time scales ($\Delta t \gg \omega_{pe}^{-1}$) the plasma is *charge neutral* — the number densities of protons and electrons will therefore balance $n_p = n_e \equiv n$. The protons are *most* of the mass so $\rho = nm_p$.

$\mathbf{v}(\mathbf{x}, t)$ Fluid velocity. $\mathbf{v}$ is the *center-of-mass velocity* of all particles within a small neighborhood of the point $\mathbf{x}$. Given the vast mismatch in masses $\mathbf{v}$ is almost always the same as the mean velocity of the protons. The electrons may move at a velocity different from the protons to produce a current $\mathbf{J}$. The momentum density of the fluid is $\rho\mathbf{v}$.

$p(\mathbf{x},t)$ Pressure. This is the sum of electron pressure and proton pressure. Any parcel of the fluid feels inward force $-p\,d\mathbf{a}$ acting on each of its outward-facing surface elements $d\mathbf{a}$. The *internal energy density* of the fluid (i.e. energy in thermal motions) is $\varepsilon = \frac{3}{2}p$.

$B(\mathbf{x},t)$ Magnetic field. This is the fundamental field in MHD. $\mathbf{J}$ and $\mathbf{E}$ are derived *from it* rather than the other way around, as is more common in electrodynamics.

MHD equations The fields described above evolve according to the governing equations which are MHD. The following is a common version of these equations[1]

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &= -\nabla\cdot(\rho\mathbf{v}) && \text{Mass continuity}\\
\rho\frac{\partial\mathbf{v}}{\partial t} + \underbrace{\rho(\mathbf{v}\cdot\nabla)\mathbf{v}}_{M.1} &= -\underbrace{\nabla p}_{M.2} + \underbrace{\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}}_{M.3} + \underbrace{\rho\mathbf{g}}_{M.4} + \underbrace{\rho\nu\nabla^2\mathbf{v}}_{M.5} && \text{Momentum}\\
\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\nabla)p &= -\gamma p(\nabla\cdot\mathbf{v}) && \text{Adiabatic gas law}\\
\frac{\partial\mathbf{B}}{\partial t} &= \underbrace{\nabla\times(\mathbf{v}\times\mathbf{B})}_{I.1} + \underbrace{\eta\nabla^2\mathbf{B}}_{I.2} && \text{Induction}
\end{aligned}
$$

Parameters of the plasma

γ The adiabatic index of the gas. Protons and electrons are point particles so $\gamma = 5/3$.

ν Kinematic viscosity. This is a diffusion coefficient, and therefore has units of cm^2/sec. According to classical kinetic theory (way beyond our scope here) viscosity arises from collisions between particles.

η The magnetic diffusivity. A diffusion coefficient (i.e. units of cm^2/sec) describing the diffusion of magnetic field through a conductor. Writing Ohm's law as $\mathbf{J} = \sigma\mathbf{E}$, the magnetic diffusivity is $\eta = c^2/4\pi\sigma$. Collisions between electrons and protons drifting past each other (at speeds much less than either thermal speed) lead to a diffusivity $\eta = 0.14\,m_e^{1/2}e^2c^2\,(k_{\rm B}T)^{-3/2}\ln\Lambda$. Ignoring the logarithmic dependence on density contained in $\ln\Lambda$, η depends only on the temperature of the plasma with which it scales inversely — hotter plasmas are less resistive.

Dimensionless numbers: Whether or not we care about a give term in the MHD equations depends on how large it is compared to other terms. Terms are compared by assigning to each field some characteristic value and to each spatial derivative the inverse of a char-

[1] Most variations of MHD involve variations in the energy equation. The adiabatic gas law is the simplest possibility. More elaborate versions can include thermal conduction, heating from viscous and resistive losses, and radiative cooling.

acteristic length L. The following ratios are common in the MHD literature

$$\begin{array}{rrcll}
\frac{M.1}{M.2}: & M^2 & \equiv & \frac{v^2}{\gamma p/\rho} = \left(\frac{v}{c_s}\right)^2 & M = \text{Mach number} \\
\frac{M.2}{M.3}: & \beta & \equiv & \frac{p}{B^2/8\pi} & \text{Plasma } \beta \\
\frac{M.1}{M.3}: & M_A^2 & \equiv & \frac{v^2}{B^2/4\pi\rho} = \left(\frac{v}{v_\mathrm{A}}\right)^2 & M_A = \text{Alfvén Mach number} \\
\frac{M.4}{M.2}: & \frac{L}{H_p} & = & \frac{L\rho}{gp} & H_p = \text{Pressure scale height} \\
\frac{M.1}{M.5}: & Re & \equiv & \frac{vL}{\nu} & \text{Reynold's number} \\
\frac{I.1}{I.2}: & Rm & \equiv & \frac{vL}{\eta} & \text{Magnetic Reynold's number}
\end{array}$$

Secondary fields which can be derived from the fundamental fields

T Temperature. Any plasma is sufficiently rare that the ideal gas law is an extremely accurate equation of state. Since the fluid's pressure consists of electron and proton pressures, both of which have density n, the ideal gas law is

$$p = 2nk_\mathrm{B}T = \frac{\rho}{\mu}k_\mathrm{B}T$$

where $\mu \simeq m_p/2$ is the average mass per particle (half the particles are electrons, and thus have almost no mass).[2]

$\mathbf{J}$ Electric current density. The electric field is sufficiently small (see below) that the *displacement current* term, $c^{-1}\partial\mathbf{E}/\partial t$, can be dropped from Ampere's law. The remaining (pre-Maxwell) equation gives the current in terms of the magnetic field

$$\mathbf{J} = \frac{c}{4\pi}\nabla\times\mathbf{B}$$

Term M.3 arises from the Lorentz for $c^{-1}\mathbf{J}\times\mathbf{B}$ after eliminating $\mathbf{J}$ using this expression for current.

$\mathbf{v}_e$ Electron velocity. On the time scales of MHD the inertia of the electrons is unimportant, so their motion is dictated by the requirement of Ampere's law. They go where the current needs them to go

$$\mathbf{v}_e = \mathbf{v} - \frac{\mathbf{J}}{en_e}$$

$\mathbf{E}$ Electric field. The plasma is charge-neutral, but Ohm's law dictates that $\mathbf{E}' = \sigma\mathbf{J}$ in the reference frame of the fluid. The laboratory-frame electric field is $\mathbf{E} = \mathbf{E}' - c^{-1}\mathbf{v}\times\mathbf{B}$ so Ohm's law is

$$\mathbf{E} = -\frac{\mathbf{v}}{c}\times\mathbf{B} + \sigma\mathbf{J}$$

The induction equation follows from taking the curl of this and using Faraday's law.

[2] When the trace elements are properly accounted for the mean mass in a fully ionized plasma of solar-system abundances is slightly higher: $\mu \simeq 0.6\,m_p$.

Frozen-in-flux

All astrophysical plasmas (and most laboratory plasmas) are hot enough and big enough that the Magnetic Reynold's number $Rm \gg 1$ and term I.2 is irrelevant.[3] Without term $I.2$ the induction equation is called the *ideal induction equation*

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) \qquad (1)$$

using a standard vector identity. While the equation appears formidable it can be interpreted quite simply: it states that every magnetic field line moves with the fluid exactly as a piece of massless thread would move. This is sometimes called the *frozen-in-flux* theorem.

The frozen-flux theorem is a statement about the relation of *trajectories* to *field lines.* A trajectory is the curve followed by an element of the fluid. The trajectory curve $\mathbf{r}(t)$ satisfies the ODE

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}[\mathbf{r}(t), t] \quad . \qquad (2)$$

This can be solved beginning at some initial time $t = t_0$ from an initial condition $\mathbf{r}(t_0) = \mathbf{r}_0$.

Magnetic field lines A field line is a space-curve which is tangent to the magnetic field $\mathbf{B}(\mathbf{x})$ at all points. The field line can be written as a vector-valued function of a scalar argument, $\mathbf{r}(\ell)$, satisfying the ODE

$$\frac{d\mathbf{r}}{d\ell} = \frac{\mathbf{B}[\mathbf{r}(\ell)]}{|\mathbf{B}[\mathbf{r}(\ell)]|} \quad . \qquad (3)$$

The scalar argument, ℓ, is length along the field line (Note that $|\partial\mathbf{r}/\partial\ell| = 1$.) The same curve may be parameterized by any scalar which is a monotonic function of position. It turns out to be convenient to work with a different parameter s defined so that $ds/d\ell = \rho/|\mathbf{B}|$. In terms of this parameter the field lines satisfies

$$\frac{d\mathbf{r}}{ds} = \frac{\mathbf{B}[\mathbf{r}(s)]}{\rho[\mathbf{r}(s)]} \qquad (4)$$

Equation (4) is formally very similar to the equation for a trajectory (2), but with $\mathbf{B}/\rho$ taking the place of $\mathbf{v}$ and s taking the place of t. Like the trajectory (4) can be solved *uniquely* beginning at any point in space $\mathbf{r}(0) = \mathbf{r}_0$ as an "initial condition". This shows that there is a unique field line passing through each and every point in space.[4]

You may have been warned that magnetic field lines are not "real". That statement means that if $\mathbf{B}$ doesn't know about the trajectories of particles (i.e. about $\mathbf{v}$) then each time you solve (4) it will yield set of field line bearing no relation to the solutions (field lines) at previous times: the apparent motion of the field lines will have nothing to do with

[3] Using the numbers provided the resistivity is $\eta \simeq 10^4\,\mathrm{cm}^2\,\mathrm{sec}^{-1}\,T_6^{-3/2}$ where T_6 is the temperature in MegaKelvins. In a one-million degree plasma $Rm = 1$ for motion at 1 m/sec (2 miles per hour) on scales of 1 meter: i.e. for a person walking! Most natural plasmas are much bigger than a person and move much faster than walking-speed so $Rm \gg 1$.

[4] There is one exception to this rule: The equation cannot be initiated at a magnetic null point, $\mathbf{B}(\mathbf{r}) = 0$. If this is used as an initial condition, the solution is a point, $\mathbf{r}(s) = \mathbf{r}_0$, rather than a curve.

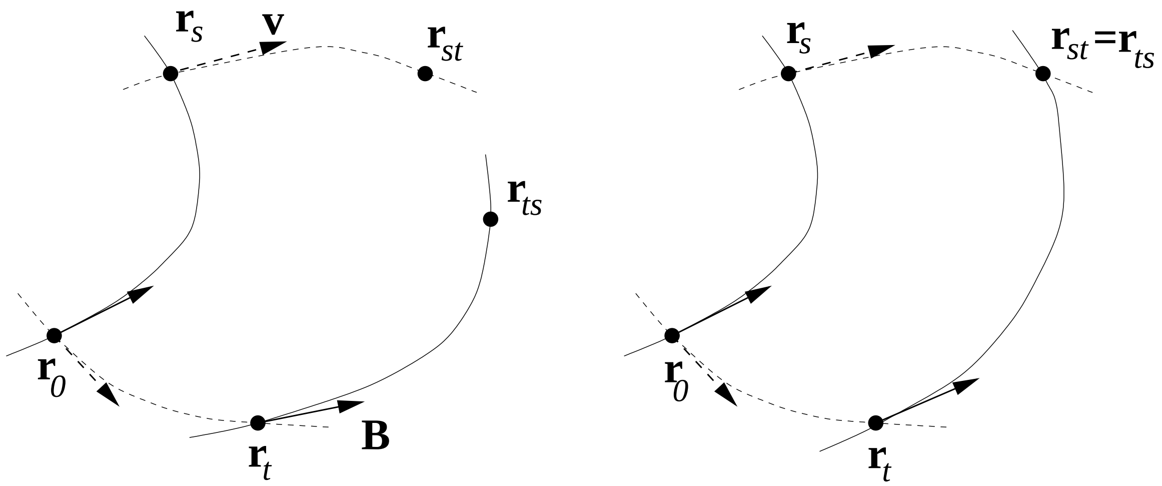

Figure 1: An illustration of the frozen-in-flux calculation. Trajectories (dashed) map $\mathbf{r}_0 \to \mathbf{r}_t$ and $\mathbf{r}_s \to \mathbf{r}_{st}$. Field lines (solid) map $\mathbf{r}_0 \to \mathbf{r}_s$ and $\mathbf{r}_t \to \mathbf{r}_{ts}$. In the left illustration there is no relation between $\mathbf{B}$ and $\mathbf{v}$ so points $\mathbf{r}_{ts} \neq \mathbf{r}_{st}$. In the right illustration the induction equation demands that $\mathbf{r}_{ts} = \mathbf{r}_{st}$.

the motion of matter — which is by definition "real". Field line are, in fact, very real in MHD because the magnetic field does know about the fluid velocity, through equation (1). This is how that equation leads to the frozen-in-flux theorem.

Consider the field line passing the point $\mathbf{r}_0$ at time t_0. One other point on that same field line is

$$\mathbf{r}_s \;\equiv\; \mathbf{r}_0 \;+\; \frac{\mathbf{B}(\mathbf{r}_0,t_0)}{\rho(\mathbf{r}_0,t_0)}\,\delta s \quad .$$

Next consider the trajectory passing $\mathbf{r}_0$ at time t_0. One other point on that trajectory is

$$\mathbf{r}_t \;\equiv\; \mathbf{r}_0 \;+\; \mathbf{v}(\mathbf{r}_0,t_0)\delta t \quad .$$

The frozen-in-flux theorem states that the material which is at point $\mathbf{r}_s$ at time t_0 moves to a point on the same field lines as $\mathbf{r}_t$. In order to demonstrate this, consider the two points $\mathbf{r}_{ts}$ and $\mathbf{r}_{st}$ which are on the field line of $\mathbf{r}_t$ and the trajectory of $\mathbf{r}_s$ respectively (see fig. 1). The first point is found by following the trajectory from $\mathbf{r}_0 \to \mathbf{r}_t$ and *then* tracing the field line defined by magnetic field $\mathbf{B}(\mathbf{x}, t_0 + \delta t)$:

$$\begin{aligned}
\mathbf{r}_{ts} \;&\equiv\; \mathbf{r}_t \;+\; \frac{\mathbf{B}(\mathbf{r}_t, t_0+\delta t)}{\rho(\mathbf{r}_t, t_0+\delta t)}\,\delta s \\
&= \overbrace{\mathbf{r}_0 \;+\; \mathbf{v}\delta t}^{\mathbf{r}_t} + \frac{\mathbf{B}}{\rho}\,\delta s + [(\mathbf{r}_t - \mathbf{r}_0)\cdot\nabla]\left(\frac{\mathbf{B}}{\rho}\right)\,\delta s + \frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right)\,\delta t\,\delta s \\
&= \mathbf{r}_0 + \mathbf{v}\delta t + \frac{\mathbf{B}}{\rho}\,\delta s + \left[\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right]\left(\frac{\mathbf{B}}{\rho}\right)\,\delta t\,\delta s
\end{aligned}$$

where $(\mathbf{r}_0, t_0)$ is implied when no explicit arguments are given.

The second point is defined by tracing the field lines from $\mathbf{r}_0 \to \mathbf{r}_s$ *and then* following the trajectory for δt:

$$
\begin{aligned}
\mathbf{r}_{st} &\equiv \mathbf{r}_s + \mathbf{v}(\mathbf{r}_s, t_0)\delta t \\
&= \overbrace{\mathbf{r}_0 + \frac{\mathbf{B}}{\rho}\,\delta s}^{\mathbf{r}_s} + \mathbf{v}\delta t + [(\mathbf{r}_s - \mathbf{r}_0)\cdot\nabla]\,\mathbf{v}\,\delta t \\
&= \mathbf{r}_0 + \frac{\mathbf{B}}{\rho}\,\delta s + \mathbf{v}\delta t + \left[\left(\frac{\mathbf{B}}{\rho}\right)\cdot\nabla\right]\mathbf{v}\,\delta t\,\delta s
\end{aligned}
$$

Forming the difference between these two points

$$
\begin{aligned}
\mathbf{r}_{ts} - \mathbf{r}_{st} &= \left\{\left[\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right]\left(\frac{\mathbf{B}}{\rho}\right) - \left[\left(\frac{\mathbf{B}}{\rho}\right)\cdot\nabla\right]\mathbf{v}\right\}\,\delta t\,\delta s \\
&= \frac{1}{\rho}\left\{\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} - \frac{\mathbf{B}}{\rho}\underbrace{\left[\frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho\right]}_{=-\rho\nabla\cdot\mathbf{v}\ \text{cont'ity}} - (\mathbf{B}\cdot\nabla)\mathbf{v}\right\}\,\delta t\,\delta s \\
&= \frac{1}{\rho}\left\{\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\right\}\,\delta t\delta s \qquad (5)
\end{aligned}
$$

The foregoing has used the definitions of *trajectory* and *field line* but assumed no relationship between $\mathbf{B}$ and $\mathbf{v}$ (it did make use of mass continuity, which is a relation between $\mathbf{v}$ and ρ). *Now* we introduce the ideal induction equation (1), which remarkably states that the entire factor in {} vanishes exactly from the final expression. Therefore the ideal induction equation implies that $\mathbf{r}_{ts} - \mathbf{r}_{st} = 0$.

This demonstrates that the ideal induction equation means that two points on the same field line at one time (namely $\mathbf{r}_0$ and $\mathbf{r}_s$) will also be on the same field line at a later time. We showed this only for points separated by infinitesimal distances over infinitesimal times. Since there is no difference at $\mathcal{O}(\delta t\,\delta s)$ we can add together (i.e. integrate) numerous separations to say the same for each and every point which shares a field line with $\mathbf{r}_0$ — and thus for the field line as a whole. Then we can add together numerous intervals to say the same about arbitrary times, and thus for all times. We therefore conclude that all material which is on the same field line will *always* be on the same field line. The field line is therefore "real" — it's made of material.[5]

An example

Consider a simple shear flow $\mathbf{v} = v_0\sin(ky)\hat{\mathbf{x}}$. For magnetic field lying in the x–y plane, $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}$ the ideal induction equation, (1), becomes

$$
\begin{aligned}
\frac{\partial B_x}{\partial t} &= -v_0\sin(ky)\frac{\partial B_x}{\partial x} \;+\; v_0 k\cos(ky)\,B_y \\
\frac{\partial B_y}{\partial t} &= -v_0\sin(ky)\frac{\partial B_y}{\partial x}
\end{aligned}
$$

[5]It is worth recalling that the original induction equation contained a diffusive term *I*.2 which was neglected to reach the ideal induction equation. Had the "non-ideal" form been placed in eq. (5) we would conclude that $\mathbf{r}_{st} \neq \mathbf{r}_{ts}$ and thus that field lines are not real at all.

If the initial configuration is a uniform vertical field $\mathbf{B}(\mathbf{x}, 0) = B_0 \hat{\mathbf{y}}$, then the solution to the induction equation is

$$\mathbf{B}(x, y, t) \;=\; B_0\, k v_0 t \cos(ky)\, \hat{\mathbf{x}} \;+\; B_0 \hat{\mathbf{y}} \quad . \tag{6}$$

Next we show that field lines from (6) are equivalent to threads dragged along with the flow. As stated above, it is possible to parameterize a field line with any monotonic function of position. Since B_y is always positive the y coordinate is just such a parameter and we can write field lines $x(y)$. The y-parameterized field line satisfies the equation

$$\frac{dx}{dy} \;=\; \frac{B_x[x(y), y]}{B_y[x(y), y]} \;=\; k v_0 t \cos(ky) \quad . \tag{7}$$

Beginning at $x(0) = x_0$ this equation has the solution

$$x(y) \;=\; v_0 t\, \sin(ky) + x_0 \quad . \tag{8}$$

Recalling the definition of velocity we can write this as $\mathbf{r}(y) \;=\; \mathbf{v}(y)\, t \;+\; \mathbf{r}_0$, which is the position of a particle moving at constant velocity for time t. This is exactly what the frozen-in-flux theorem said would happen: each bit of the field line moves with the fluid.

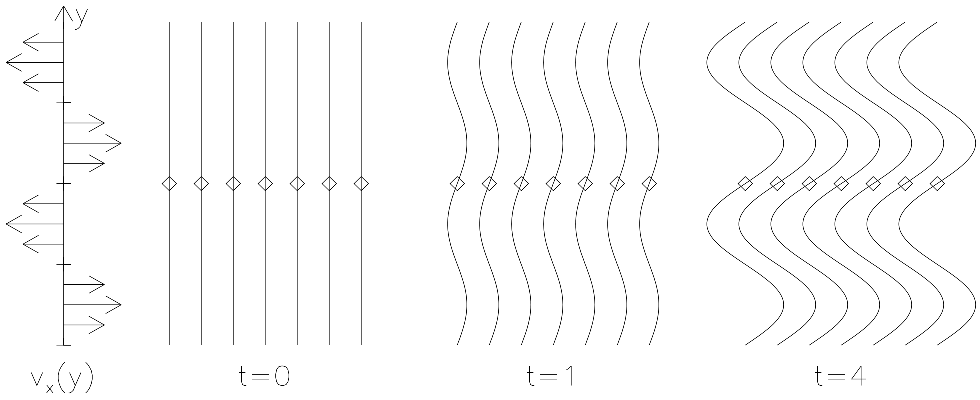

Figure 2: The shear flow (left) and some field lines of the evolving magnetic field. The initial points for these field lines (diamonds) lie along $y = 0$ where the velocity vanishes.

Magnetic Pressure and Magnetic Tension

The induction equation describes how the magnetic field changes under a given flow field $\mathbf{v}(\mathbf{x}, t)$. Neglecting magnetic diffusivity it states that field lines will flow with the material. To know how the material flows, however, we must solve the momentum equation for $\mathbf{v}(\mathbf{x}, t)$. This equation depends on the magnetic field through the term $M.3$: $\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$. In light of this interdependence the full set of MHD equations must be solved simultaneously — $\mathbf{B}$ depends on $\mathbf{v}$ while $\mathbf{v}$ also depends on $\mathbf{B}$.

High-β case: The interdependency problem goes away in those cases where term $M.3$ may be dropped from the momentum equation — i.e. where the magnetic forces are negligible. Referring to the table of dimensionless ratios we see that $M.3$ may be dropped whenever $\beta \gg 1$ (the high-β limit). Doing so reduces the momentum equation to that of traditional fluid dynamics, called the Navier-Stokes equations. In this system the fluid motion is dictated by pressure, inertia, gravity and possibly by viscosity. This is not to say that the Navier-Stokes equations are easy to solve, but since we've lived our entire lives within a non-conducting fluid (the air) we can envision what solutions might be like.

Once we solve these equations within a region and time-interval of interest we will know the flow velocity $\mathbf{v}(\mathbf{x}, t)$. We can now use this known function in the induction equation and solve for the magnetic field $\mathbf{B}(\mathbf{x}, t)$ beginning with the initial condition $\mathbf{B}(\mathbf{x}, 0)$. To envision this, think of tracing many field lines of the initial magnetic field. We can try to choose initial positions so that the transverse spacing of lines will be inversely related to the field strength $|\mathbf{B}|$ (not automatically the case!). If we "paint" these into the fluid with dye, then the dye-lines will be moved exactly like the field lines, according to the frozen-in-flux theorem. Dye is a classic example of a "passive field": it is affected by the flow but does not affect the flow in return. In the high-β limit the magnetic field is also a passive field.

Magnetic Forces In general cases we are faced with equations of fluid dynamics which are modified by the presence of a force density due to the magnetic field (called the Loretnz force). The force density is

$$\mathbf{f}^{(\mathrm{mag})} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\underbrace{\nabla \frac{|\mathbf{B}|^2}{8\pi}}_{\text{pressure}} + \underbrace{\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\text{tension}} \tag{9}$$

where the final expression exploits a vector calculus identity. The first term on the right can be re-cast to look like the traditional hydrodynamic pressure force $-\nabla p_{\mathrm{M}}$ where

$$p_{\mathrm{M}} \equiv \frac{|\mathbf{B}|^2}{8\pi} \quad ,$$

is often called the magnetic pressure.

If there were only a magnetic pressure term then a magnetic field would affect the plasma exactly like another phase of the fluid, with its own pressure. There is also the tension term. Rewriting $\mathbf{B} = B\hat{\mathbf{b}}$ the tension term is

$$\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} = \hat{\mathbf{b}}\,\hat{\mathbf{b}}\cdot\nabla\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}} \quad .$$

The first term here is equal and opposite to that component of the pressure term acting along $\hat{\mathbf{b}}$. When this is added to $-\nabla p_{\mathrm{M}}$ we get a pressure which acts only perpendicular to the magnetic field — a strange gas indeed. This is re-assuring because expression we began with, $\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$ is obviously perpendicular to $\mathbf{B}$ and thus to $\hat{\mathbf{b}}$.

To understand the last term we return to the length-parameterized field line expression (3), whose right-hand-side is the unit vector $\hat{\mathbf{b}}$. We also note that $\hat{\mathbf{b}}\cdot\nabla = \partial/\partial\ell$, is the directional derivative along a field line. This makes the remaining term

$$\frac{B^2}{4\pi}(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}} = \frac{B^2}{4\pi}\frac{\partial^2\mathbf{r}}{\partial\ell^2} = \frac{B^2}{4\pi}\mathbf{k}$$

where $\mathbf{k}$ is called the curvature vector in the mathematics of space-curves; it's magnitude is the inverse of the local radius of curvature and it points toward the center of curvature. When a string under tension T is bent it is subject to a linear force density (force per unit length) $T\mathbf{k}$. Comparing this to the expression above it seems that a bundle of field lines behaves like a string with tension $T = AB^2/4\pi$, where A is the area of the bundle. A wave travels along the string at speed $v_\phi = \sqrt{T/\mu}$ where μ is the linear mass density. The linear mass density of the field line bundle is $\mu = A\rho$. By analogy to a string, the field lines will support waves with phase speed

$$v_{\mathrm{A}} \;=\; \frac{B}{\sqrt{4\pi\rho}} \quad , \tag{10}$$

known as the Alfvén speed.

To summarize, the magnetic force density on the plasma

$$\mathbf{f}^{(\mathrm{mag})} \;=\; -\nabla_{\perp} p_{\mathrm{M}} \;+\; \frac{B^2}{4\pi}\,(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}} \tag{11}$$

consists of a magnetic pressure gradient, which acts only perpendicular to the field, and a magnetic tension (which is naturally perpendicular to the field).

The Lorentz force is actually a generic feature of Maxwell's equations. Writing it in index notation gives

$$f_i^{(\mathrm{mag})} \;=\; \frac{\partial}{\partial x_j}\left(\frac{B_iB_j}{4\pi} \;-\; \frac{B^2}{8\pi}\delta_{ij}\right) \;=\; \frac{\partial T_{ij}}{\partial x_j} \tag{12}$$

where T_{ij} is the Maxwell stress tensor for the case $\mathbf{E} = 0$. Everything here has been in cgs units. The SI version of the expression may be recovered by replacing factors of 4π with μ_0 the permeability of free space.[6] On the basis of our previous discussion we see that the Maxwell stress tensor consists of an isotropic pressure term $-B^2/8\pi$ and an off-diagonal term capable of creating shear shear stresses — this is the magnetic tension.

Example

The magnetic field from the induction example (6) is

$$\mathbf{B}(x,y,t) \;=\; B_0\,kv_0t\cos(ky)\,\hat{\mathbf{x}} \;+\; B_0\hat{\mathbf{y}} \quad . \tag{13}$$

The force-density may be readily found

$$\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} \;=\; \frac{1}{4\pi}\,(B_0v_0t)^2k^3\sin(ky)\cos(ky)\hat{\mathbf{y}} \;-\; \frac{1}{4\pi}\,B_0^2k^2v_0t\sin(ky)\hat{\mathbf{x}}$$

The first term is a vertical force directed away from regions of stronger magnetic field: $ky = m\pi$ — the is the pressure force. The second term acts horizontally, and is strongest where the field is most curved $ky = \pm\pi/2$, $\pm 3\pi/2$. Referring to fig. 2 we see that the force

[6] In fact, the actual value is $\mu_0 = 4\pi\times 10^{-7}$, where the factor 10^{-7} can be thought of as a conversion from Gauss^2 to Tesla^2 (10^{-8}) and from J m^{-3} to erg cm^{-3} ($\times 10$). Often astrophysicists use a kind of hybrid-SI system in which $\mathbf{B}$ is measured in Gauss and lengths and energies are measured in cm and ergs, but Ampere's law is written with a μ_0. In this case $\mu_0 = 4\pi$.

opposes the bending of the field lines — it is leftward at $ky = \pi/2$, rightward at $ky = -\pi/2$ etc. This is the tension force. To see this relation mathematically we can re-write the above equation as

$$\begin{aligned} \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} &= -\frac{1}{8\pi}\hat{\mathbf{y}}\frac{\partial}{\partial y}\left[B_0 v_0 kt\cos(ky)\right]^2 \;+\; \frac{1}{4\pi}B_0\frac{\partial}{\partial y}\left[B_0 k v_0 t\cos(ky)\hat{\mathbf{x}}\right] \\ &= -\frac{1}{8\pi}\nabla|\mathbf{B}|^2 \;+\; \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} \end{aligned}$$

To work self-consistently we replace $v_0 t\sin(ky) = x(y,t)$ and discard terms proportional to x^2. This leaves only the tension force

$$\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \;\simeq\; -\frac{B_0^2 k^2}{4\pi}\, x(y)\,\hat{\mathbf{x}} \;=\; \frac{B_0^2}{4\pi}\frac{\partial^2 x}{\partial y^2}\,\hat{\mathbf{x}} \tag{14}$$

The horizontal velocity is $\mathbf{v} = \hat{\mathbf{x}}\partial x/\partial t$ and the horizontal momentum equation becomes

$$\rho\,\frac{\partial^2 x}{\partial t^2} \;=\; \frac{B_0^2}{4\pi}\,\frac{\partial^2 x}{\partial y^2}\quad , \tag{15}$$

which is the wave equation. This has the periodic solution

$$x(y,t) \;=\; A\,\sin(ky)\,\sin(k v_{\rm A} t) \;=\; \tfrac{1}{2}A\cos[k(y - v_{\rm A}t)] + \tfrac{1}{2}A\cos[k(y + v_{\rm A}t)] \quad . \tag{16}$$

This is a standing Alfvén wave of wavelength $2\pi/k$. In the final expression it is decomposed into counter-propagating traveling waves.

This scenario is similar to every other oscillation or wave in Physics: motion leads to a restoring force, which reverses the motion, overshoots and leads to the opposite restoring force. Here the shear flow drags the field lines as depicted in fig. 2, according to the frozen-in-flux theorem. The distorted field produces a restoring force, magnetic tension, which opposes the shear flow, eventually reversing it. The reversed flow then undoes the distortion (fig. 2 form right to left), overshooting and creating the opposite distortion. Thus the Alfvén wave is one in which magnetic tension provides the restoring force; its phase speed is $v_{\rm A} = B/\sqrt{4\pi\rho}$.

There are two other waves in MHD: the fast magnetosonic (FMS) wave and the slow magnetosonic (SMS) wave. The restoring forces in the FMS wave is the combined pressures of the plasma and the field, $p + p_{\rm M}$. The restoring force for the SMS wave is the plasma pressure alone, p, just like the accoustic wave (sound wave) in the Navier-Stokes equation. The magnetic pressure is absent from this case because the gradients are in the direction of $\mathbf{B}$, where the magnetic pressure exerts no force. It resembles a sound wave in a hose.