



Angular  
Momentum

# The Vector Product

*a.k.a. The Cross Product*

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{c}}$$

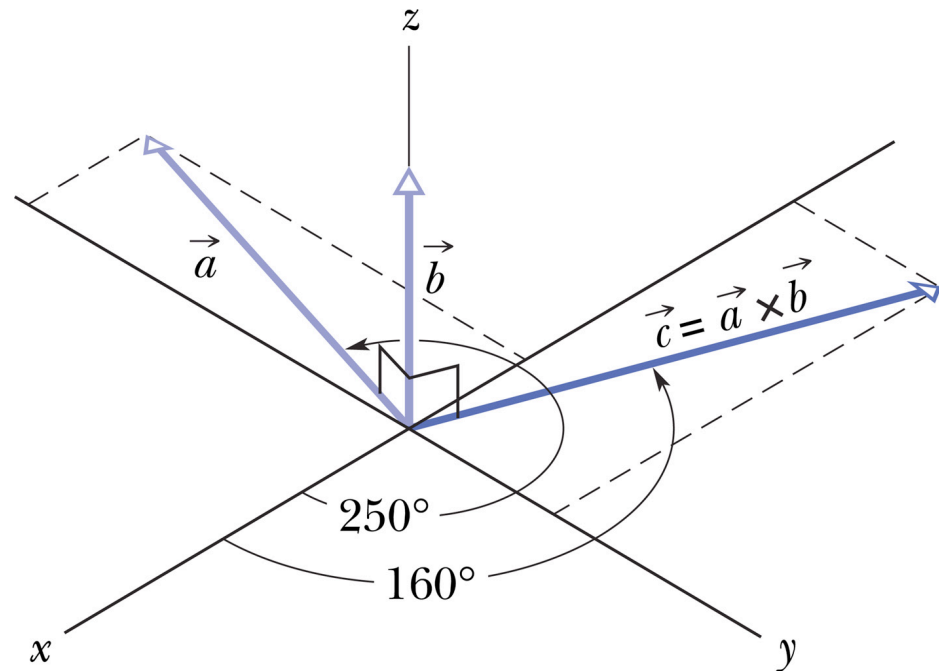
multiply two vectors...  
get a vector

magnitude:  $|\vec{\mathbf{c}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\phi$

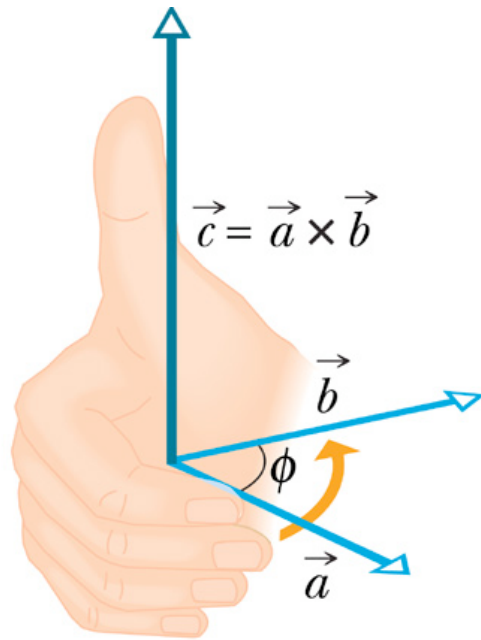
direction:

$\vec{\mathbf{a}}$  and  $\vec{\mathbf{b}}$  lie in a plane.\*  
Their cross product  $\vec{\mathbf{c}}$   
is a vector per-  
pendicular (normal) to  
that plane

\*The cross product is really a  
math-trick for representing a  
plane by a vector



There are 2 directions perpendicular to any plane. Which one do we choose?

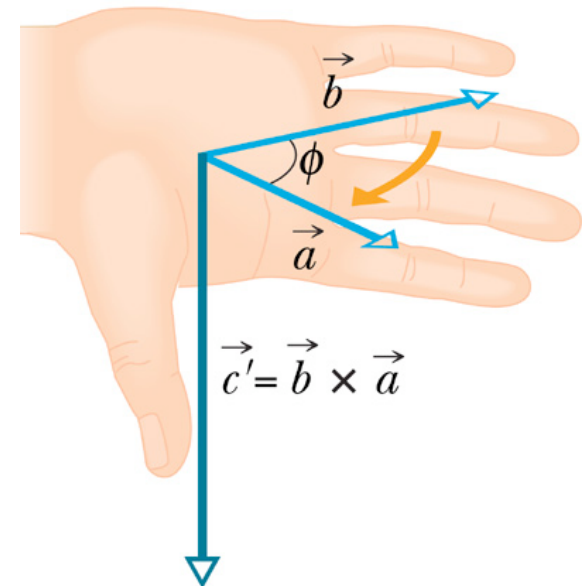


**The right\* hand rule:**

thumbs shows which side of plane to use. (chooses between 2 options)

Note: order of multiplication matters:

$$\vec{\mathbf{b}} \times \vec{\mathbf{a}} = -(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$



\*a convention: like driving on the right

# Cross product


like other kinds of multiplication you know ...

$$\vec{\mathbf{a}} \times (3\vec{\mathbf{b}}) = 3(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = (3\vec{\mathbf{a}}) \times \vec{\mathbf{b}}$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \times \vec{\mathbf{c}}) \quad \text{distributive}$$

except that **order matters** (non-commutative):

$$\vec{\mathbf{b}} \times \vec{\mathbf{a}} = -(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

  $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = 0$

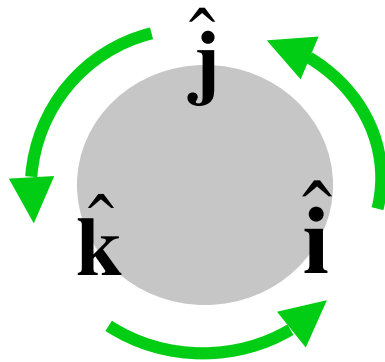
With component representation:

$$\left. \begin{aligned} \vec{\mathbf{a}} &= a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}} \\ \vec{\mathbf{b}} &= b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}} + b_z \hat{\mathbf{k}} \end{aligned} \right\} \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

brute force approach. best when  
no component is zero

## With component representation:

know cross products of unit vectors:



$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

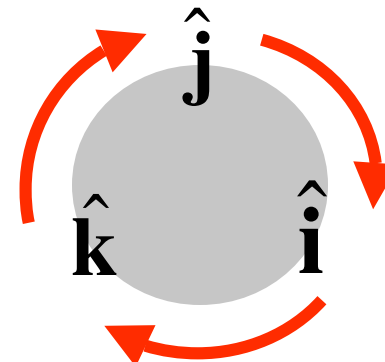
**duplicate**

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

**cyclic**



$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

**counter-cyclic**

$$(a_x \hat{\mathbf{i}} + a_z \hat{\mathbf{k}}) \times (b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}}) =$$

$$= a_x b_x (\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + a_x b_y (\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + a_z b_x (\hat{\mathbf{k}} \times \hat{\mathbf{i}}) + a_z b_y (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$$
$$= a_x b_y \hat{\mathbf{k}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$$

$$\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

$$\vec{\mathbf{b}} = 2\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = ?$$

A  $6\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

B  $4\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$

C  $-4\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$

D  $4\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

E  $-4\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

$$\vec{a} = 3\hat{i} - 2\hat{j}$$

$$\vec{b} = 2\hat{k}$$

$$\vec{a} \times \vec{b} = ?$$

A  $6\hat{i} - 4\hat{j}$

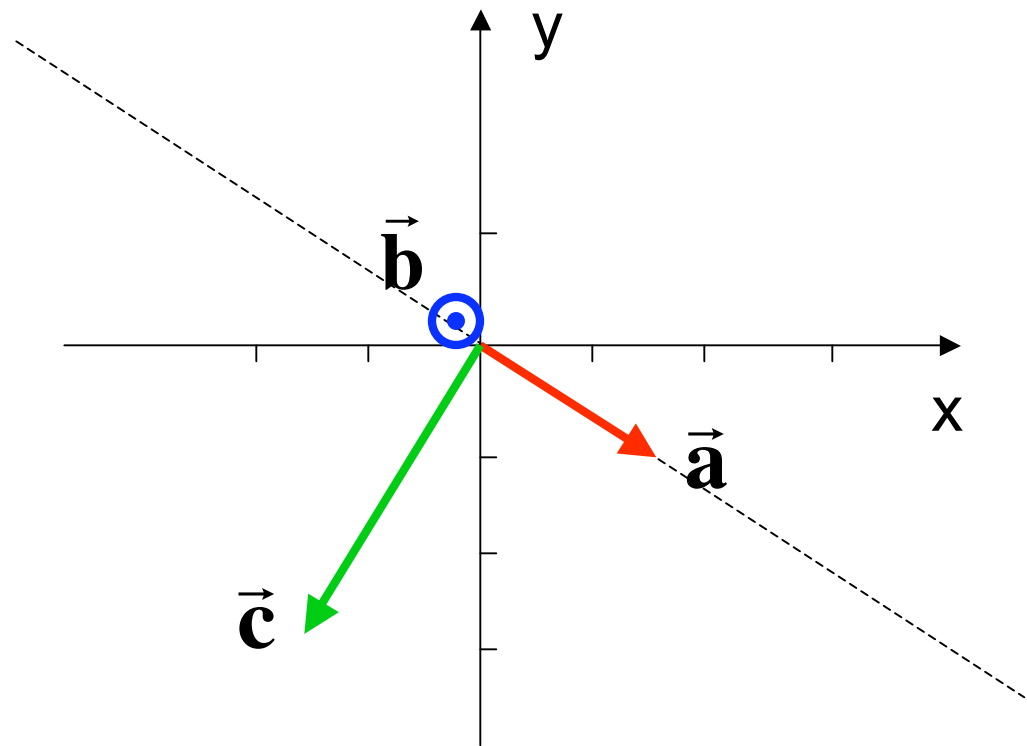
B  $4\hat{i} - 6\hat{j}$

C  $-4\hat{i} - 6\hat{j}$

D  $4\hat{i} + 6\hat{j}$

E  $-4\hat{i} + 6\hat{j}$

$$\begin{aligned}(3\hat{i} - 2\hat{j}) \times (2\hat{k}) &= 6(\hat{i} \times \hat{k}) - 4(\hat{j} \times \hat{k}) \\ &= 6(-\hat{j}) - 4(\hat{i})\end{aligned}$$



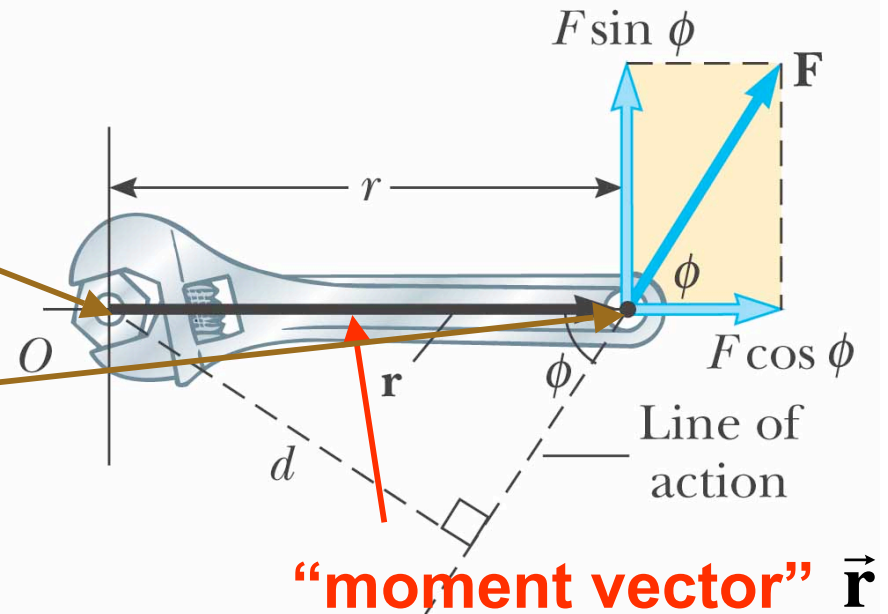
# Forms of Newton's 2<sup>nd</sup> Law

Linear	Rotational
$\sum \vec{\mathbf{F}} = m \vec{\mathbf{a}}_{\text{cm}}$	$\sum \vec{\tau} = I \vec{\alpha}$
$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$	

# Computing torque in 3d

1. Torque is computed about a given **point**<sup>1</sup>

2. One must know **where** force is applied



<sup>1</sup> point will lie on axis of rotation

# Computing torque in 3d

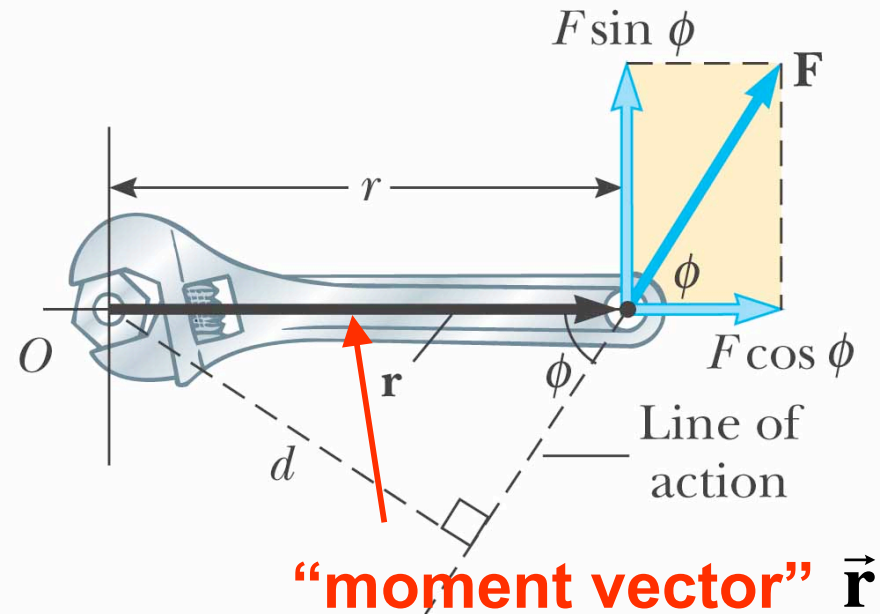
1. Torque is computed about a given point<sup>1</sup>
2. One must know **where** force is applied

torque is a vector\*\*:

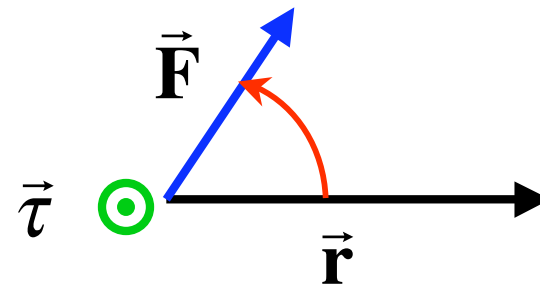
$$\vec{\tau} = \vec{r} \times \vec{F}$$

<sup>1</sup> point will lie on axis of rotation

\*\*torque occurs w/in plane



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1. every torque is computed about same point.
2. add vectors head-to-tail

$$\sum \vec{\tau} = I \vec{\alpha}$$

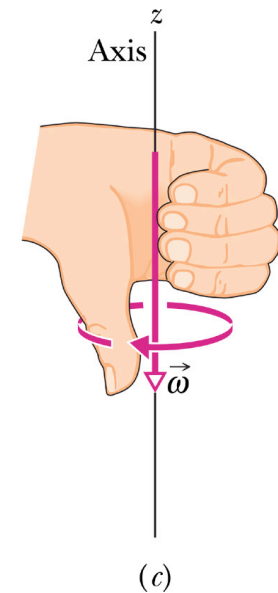
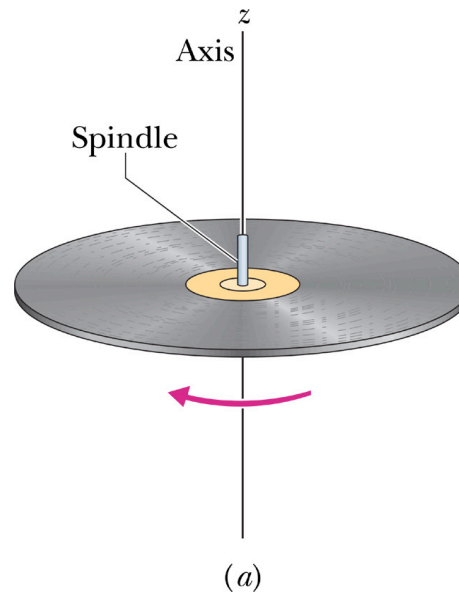
angular velocity  $\vec{\omega}$  is also  
a vector:

**magnitude:**  $| d\theta/dt |$  [ rad/s ]

**direction:** along axis  
of rotation

BUT which way along  
axis? (there are 2  
options)

decide using right-  
hand rule\*

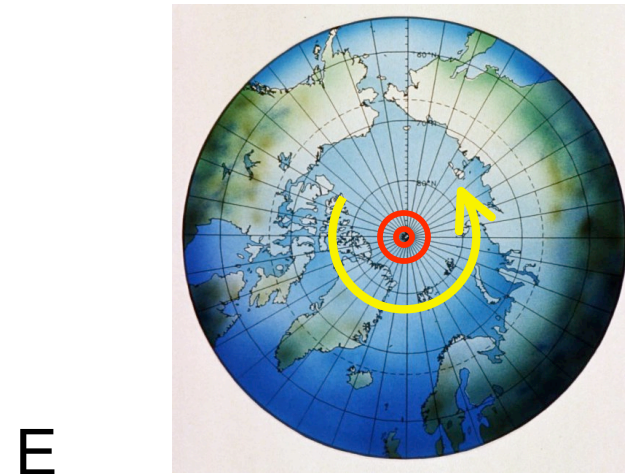
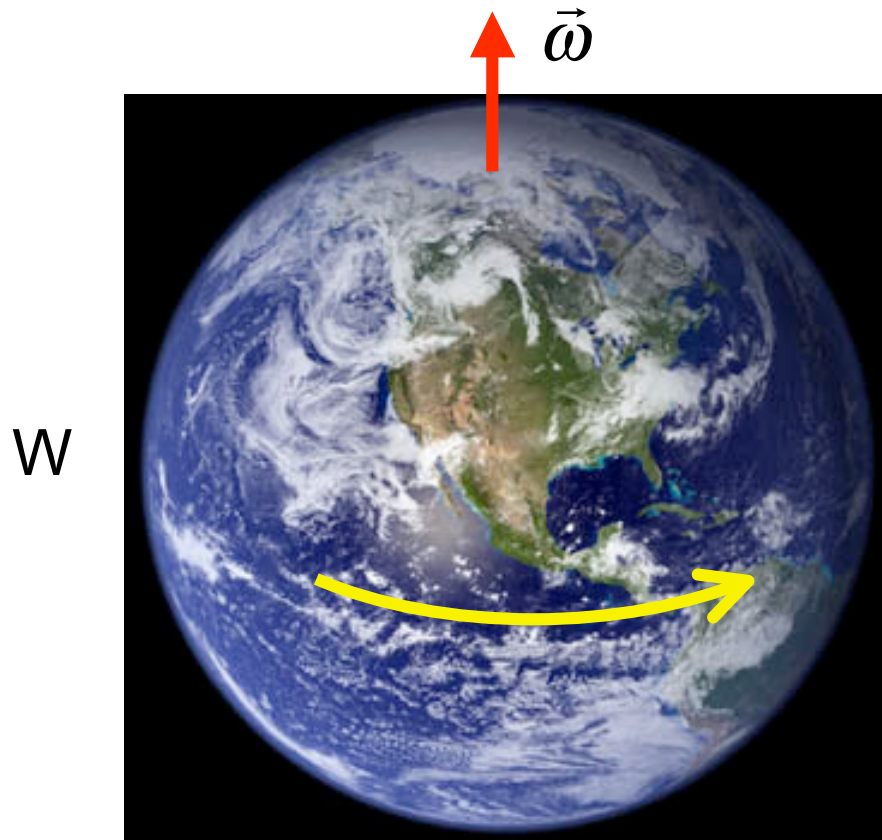


\* this is sort of different version. the "curly"  
RHR. cross products use the "wavy" RHR.



The angular rotation vector of the Earth is directed:

- A** From East to West
- B** From West to East
- C** From South Pole to North Pole
- D** From North Pole to South Pole



Sun rises first on the East Coast: NY passes under the Sun first.

The angular rotation vector of the Earth is directed:

A From East to West

C From South Pole to North Pole

B From West to East

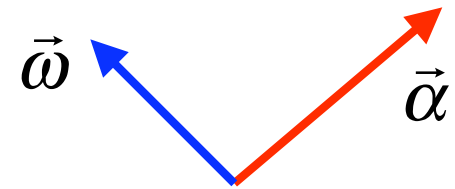
D From North Pole to South Pole

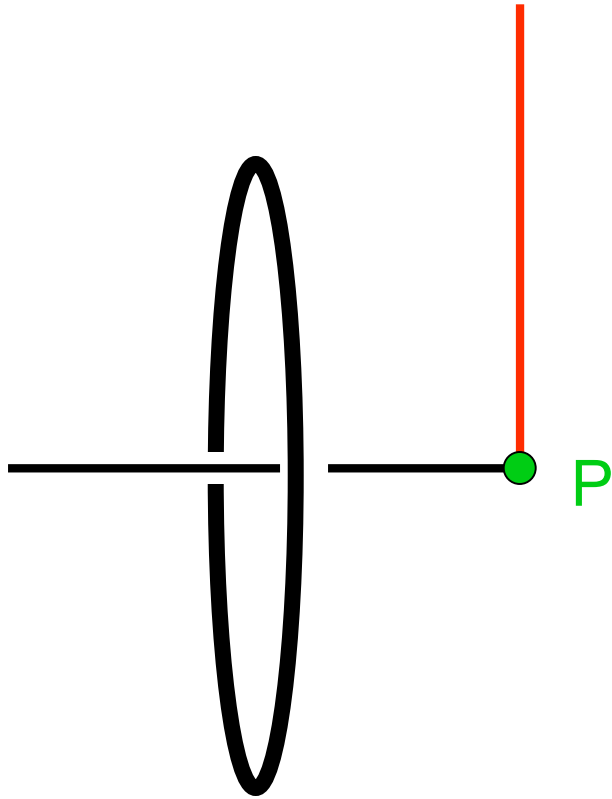
angular  
acceleration is  
a vector

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d\omega_x}{dt} \hat{\mathbf{i}} + \frac{d\omega_y}{dt} \hat{\mathbf{j}} + \frac{d\omega_z}{dt} \hat{\mathbf{k}}$$

If  $\vec{\alpha}$  is perpendicular to  $\vec{\omega}$  it  
means

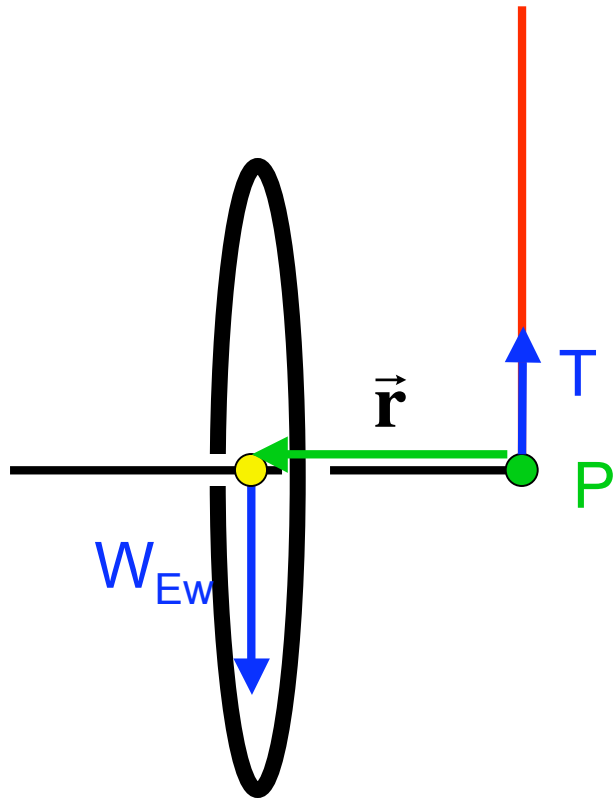
- A The spinning is increasing
- B The spinning is decreasing
- C The axis of rotation is changing**
- D That cannot happen





The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. At the instant the axle is horizontal the net torque about point **P** is directed:

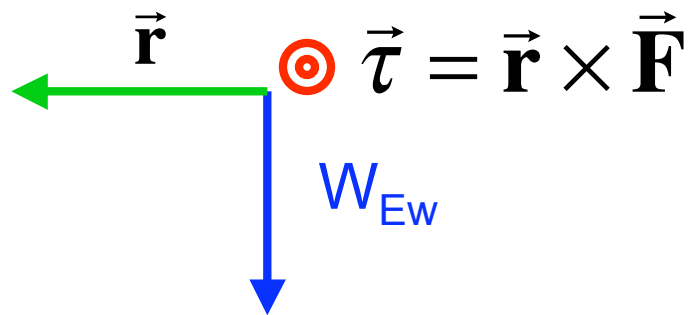
- A** Toward you
- B** Away from you
- C** Upward
- D** Downward
- E** To the left

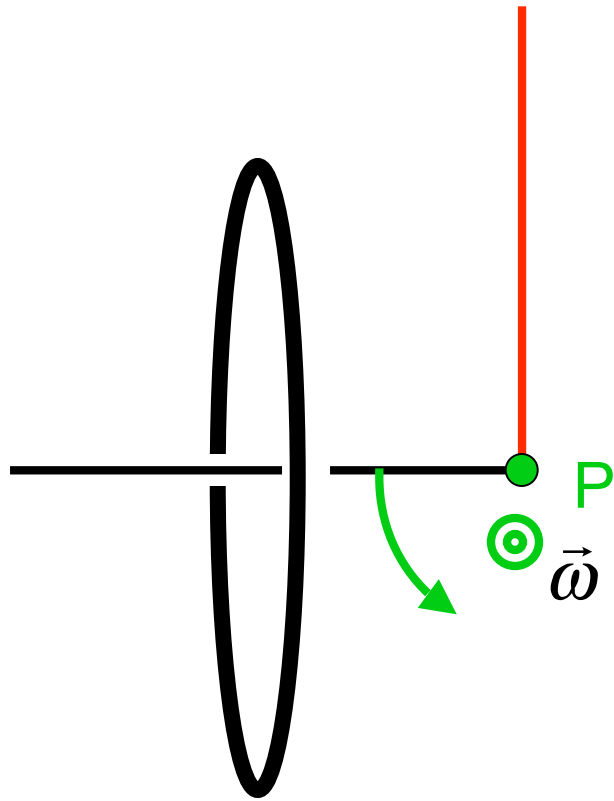


torque from T:  $\tau = 0$

The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. At the instant the axle is horizontal the net torque about point P is directed:

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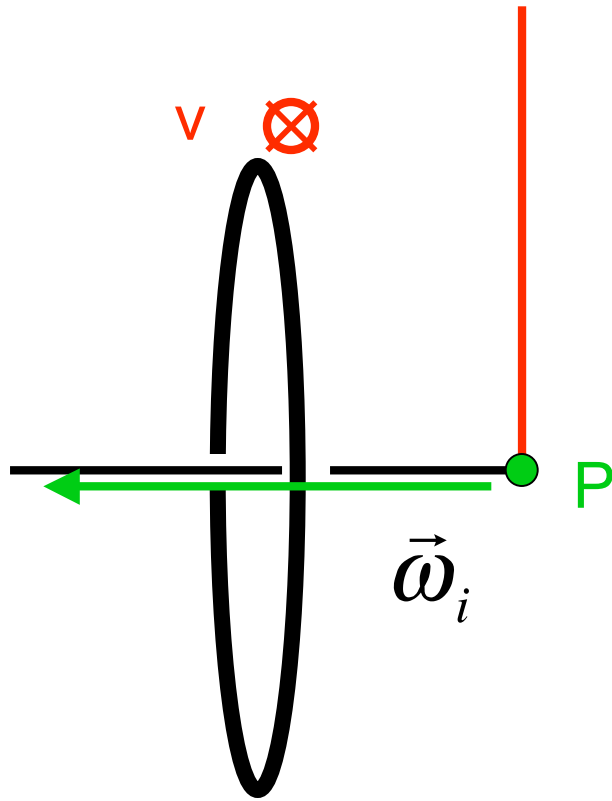
The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is stationary**. Soon after this time the **angular velocity** will be directed

- A Toward you
- B Away from you
- C Downward
- D To the left
- E To the right

$\vec{\tau} = I\vec{\alpha}$

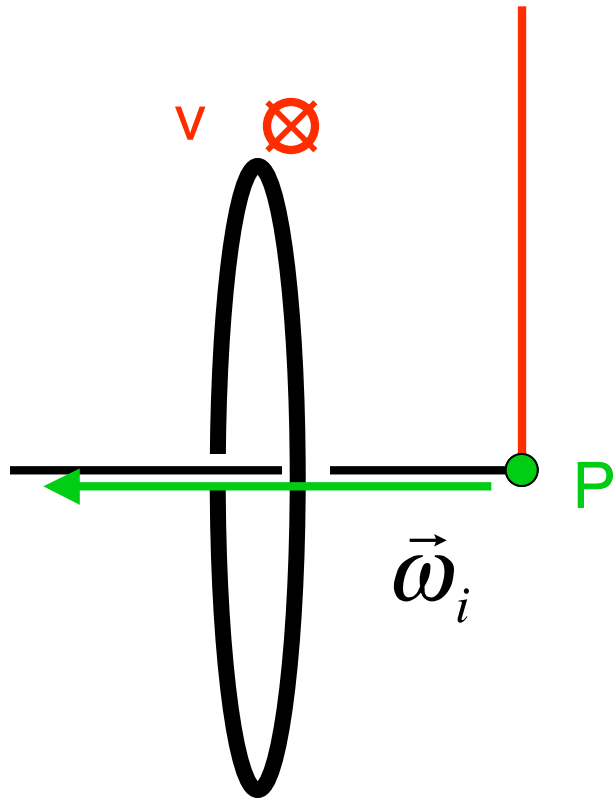
$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

$\vec{\omega}_i = 0$



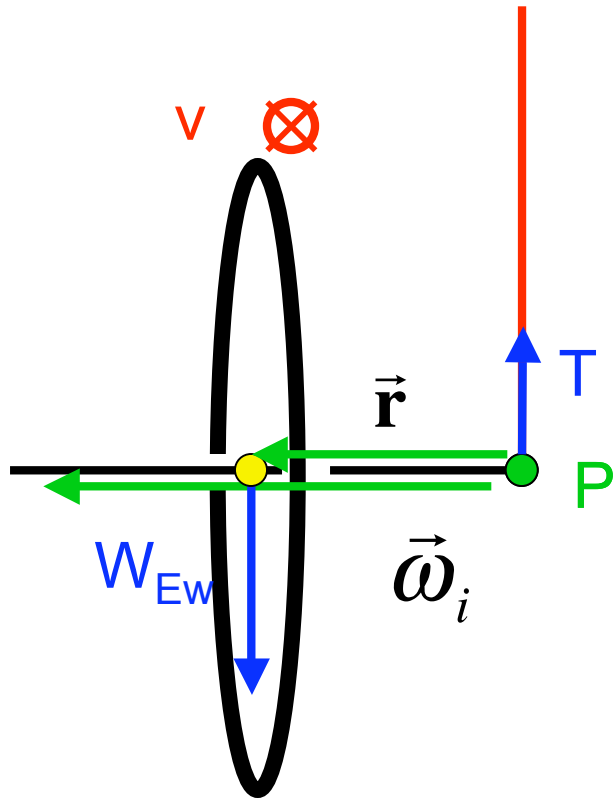
The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is spun about the axle so the top moves away from you**. At this time the angular velocity is directed

- A Toward you
- B Away from you
- C Downward
- D To the left**
- E To the right



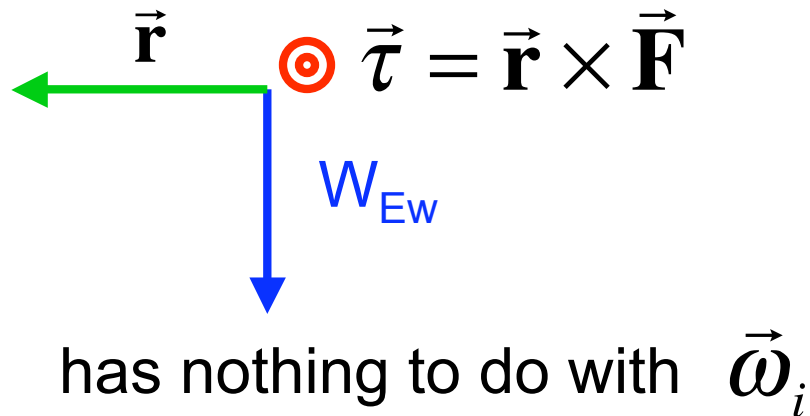
The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is spun about the axle so the top moves away from you**. Now the torque about P is directed

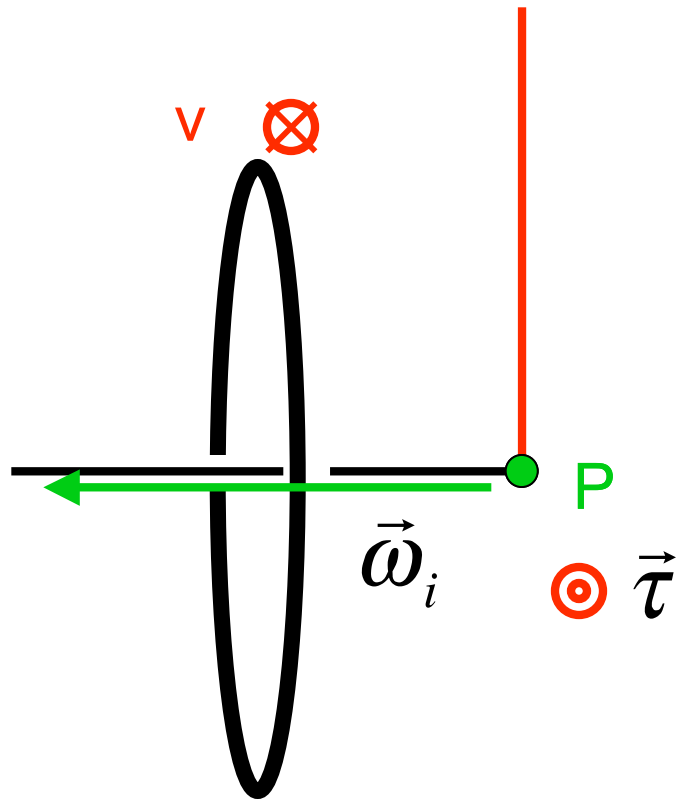
- A Toward you
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- D To the left
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The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is spun about the axle so the top moves away from you**. Now the torque about P is directed

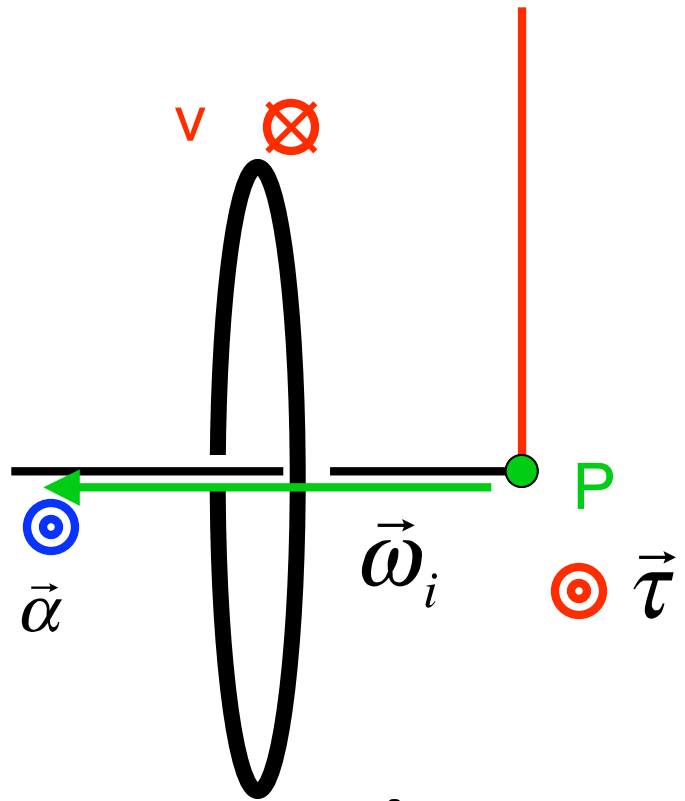
- A Toward you
- B Away from you
- C Downward
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- E To the right





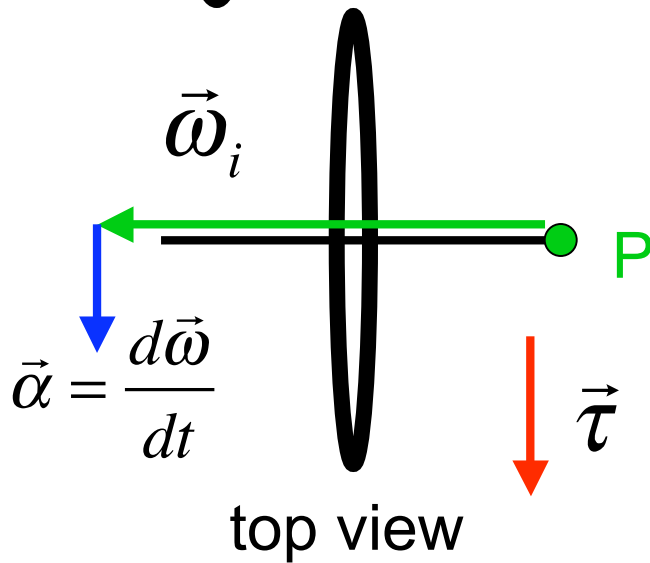
The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is spun about the axle so the top moves away from you**. The angular velocity vector will change by:

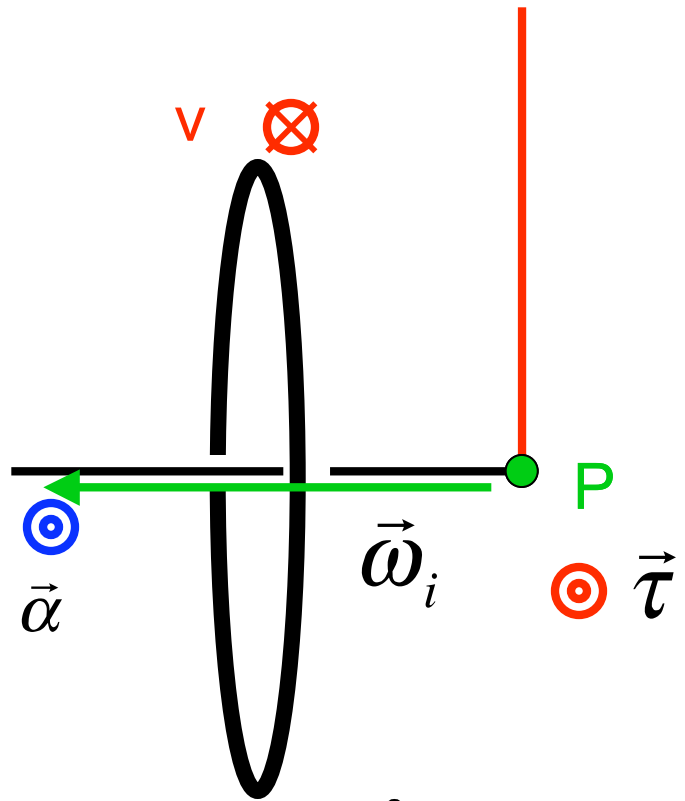
- A Moving toward you
- B Moving away from you
- C Moving downward
- D Increasing magnitude
- E Decreasing magnitude



The axle of a bicycle wheel is tied to a light string. Nothing else supports the wheel. Initially the axle is horizontal and **the wheel is spun about the axle so the top moves away from you**. The angular velocity vector will change by:

- A Moving toward you
- B Moving away from you
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- D Increasing magnitude
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- A Moving toward you
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