

MATHEMATICAL FORMULAE:

Trigonometric functions

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	3π	4π	$n\pi$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	+1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	0	0	0
$\cos(x)$	+1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	+1	-1	+1	$(-1)^n$

$$\sin^2 x + \cos^2 x = 1 \quad , \quad 1 + \tan^2 x = \sec^2 x \quad , \quad 1 + \cot^2 x = \csc^2 x$$

$$e^{ix} = \cos x + i \sin x \quad , \quad \cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad , \quad \sin(x) = -i\frac{1}{2}(e^{ix} - e^{-ix})$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad , \quad \sin(2a) = 2 \sin a \cos a$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1$$

Hyperbolic functions

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad , \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \quad , \quad e^x = \cosh(x) + \sinh(x)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad , \quad 1 - \tanh^2 x = \operatorname{sech}^2 x \quad ,$$

$$\frac{d \cosh(x)}{dx} = \sinh(x) \quad , \quad \frac{d \sinh(x)}{dx} = \cosh(x) \quad , \quad \frac{d \tanh(x)}{dx} = \operatorname{sech}^2(x) \quad ,$$

$$\cos(ix) = \cosh(x) \quad , \quad \sin(ix) = i \sinh(x) \quad , \quad \tan(ix) = i \tanh(x)$$

Indefinite Integrals (anti-derivatives)

$$\int \sin(x) \cos(x) dx = -\frac{1}{4} \cos(2x)$$

$$\int \sin^2(x) dx = \frac{1}{2}x - \frac{1}{4} \sin(2x) \quad , \quad \int \cos^2(x) dx = \frac{1}{2}x + \frac{1}{4} \sin(2x)$$

$$\int x \sin(ax) dx = \frac{1}{a^2} \sin(ax) - \frac{x}{a} \cos(ax)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) \quad , \quad \int \frac{dx}{1+x^2} = \tan^{-1}(x) \quad ,$$

$$\int \frac{x dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} \quad , \quad \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$$

$$\int \frac{x dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2+a^2}} \quad , \quad \int \frac{x dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

Definite Integrals (m, n are integers > 0)

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{1}{2}a\delta_{mn} \quad , \quad \int_0^a \sin^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{8}[2 + \delta_{mn}]$$

$$\int_{-\infty}^{\infty} e^{-u^2/2} du = \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du = \sqrt{2\pi} \quad , \quad \int_0^{\infty} u^n e^{-\alpha u} du = \alpha^{-n-1} n!$$

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} \quad .$$

Sums & Series:

$$\frac{1-x^n}{1-x} = \sum_{j=0}^n x^j = 1 + x + \cdots + x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \cdots \quad , \quad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \quad , \quad \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 \cdots + \frac{(2n)!}{4^n (n!)^2} x^n \cdots$$

Fourier transforms:

$$\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\delta(k) \quad , \quad \int_{-\infty}^{\infty} e^{-x^2/\lambda^2} e^{ikx} dx = \sqrt{\pi}\lambda e^{-k^2\lambda^2/4}$$

$$\int_{-\infty}^{\infty} e^{-|x|/\lambda} e^{ikx} dx = \frac{2\lambda}{1+\lambda^2 k^2} \quad , \quad \int_{-\infty}^{\infty} \operatorname{sech}(x/\lambda) e^{ikx} dx = 2\lambda \operatorname{sech}(\lambda k/2)$$

$$\int_{-\infty}^{\infty} \operatorname{Ai}(x/\lambda) e^{ikx} dx = \lambda e^{-i\lambda^3 k^3/3} \quad , \quad \frac{1}{\pi} \int_0^{\pi} e^{iz \cos \theta} \cos(n\theta) d\theta = i^n J_n(z)$$

Miscellaneous

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad , \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vector Calculus

Vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \quad , \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad ,$$

$$\nabla \cdot (f\mathbf{a}) = \nabla f \cdot \mathbf{a} + f\nabla \cdot \mathbf{a} \quad , \quad \nabla \times (f\mathbf{a}) = \nabla f \times \mathbf{a} + f\nabla \times \mathbf{a}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

Cylindrical Coordinates: (r, ϕ, z) , $x = r \cos \phi$, $y = r \sin \phi$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \quad , \quad d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}} \quad , \quad dV = r dr d\phi dz \quad ,$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = \left[\frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{r}} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{\boldsymbol{\phi}} + \left[\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi} \right] \frac{\hat{\mathbf{z}}}{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \quad , \quad \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates: (r, θ, ϕ) , $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}} \quad , \quad dV = r^2 \sin \theta dr d\theta d\phi \quad ,$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\nabla \times \mathbf{v} = \left[\frac{\partial(\sin \theta v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right] \frac{\hat{\mathbf{r}}}{r \sin \theta} + \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial(rv_\phi)}{\partial r} \right] \frac{\hat{\boldsymbol{\theta}}}{r} + \left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] \frac{\hat{\boldsymbol{\phi}}}{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Special functions

Bessel functions — $\Omega_m(x)$ stands for either $J_m(x)$ or $Y_m(x)$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) - \frac{m^2}{r^2} f = -k^2 f \implies f(r) = J_m(kr) \text{ or } f(r) = Y_m(kr)$$

$$J_{-m}(x) = (-1)^m J_m(x) \quad , \quad m \text{ integer}$$

$$\Omega_{m-1}(x) + \Omega_{m+1}(x) = \frac{2m}{x} \Omega_m(x) \quad , \quad \Omega_{m-1}(x) - \Omega_{m+1}(x) = 2\Omega'_m(x)$$

$$J_m(x) = \frac{1}{m!} \left(\frac{x}{2} \right)^m + \dots \quad , \quad x \ll 1$$

$$Y_0(x) = \frac{2}{\pi} \ln(x) + \dots \quad , \quad Y_m(x) = -\frac{(m-1)!}{\pi} \left(\frac{x}{2} \right)^{-m} + \dots \quad , \quad m \geq 1$$

$$\int_0^1 J_m(j_{m,n}x) J_m(j_{m,n'}x) x dx = \frac{1}{2} J_{m+1}^2(j_{m,n}) \delta_{n,n'} = \frac{1}{2} [J'_m(j_{m,n})]^2 \delta_{n,n'} \quad ,$$

Zeroes of $J_m(x)$ — $J_m(j_{m,n}) = 0$

n	$j_{0,n}$	$J_1(j_{0,n})$	$j_{1,n}$	$J_2(j_{1,n})$	$j_{2,n}$	$J_3(j_{2,n})$	$j_{3,n}$	$J_4(j_{3,n})$	$j_{4,n}$	$J_5(j_{4,n})$
0	2.40	0.519	3.83	0.403	5.14	0.340	6.38	0.298	7.59	0.268
1	5.52	-0.340	7.02	-0.300	8.42	-0.271	9.76	-0.249	11.06	-0.232
2	8.65	0.271	10.17	0.250	11.62	0.232	13.02	0.218	14.37	0.206
3	11.79	-0.232	13.32	-0.218	14.80	-0.207	16.22	-0.196	17.62	-0.188

Modified Bessel functions

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) - \frac{m^2}{r^2} f = +k^2 f \implies f(r) = I_m(kr) \text{ or } f(r) = K_m(kr)$$

$$I_m(x) = \frac{1}{m!} \left(\frac{x}{2} \right)^m + \dots \quad , \quad x \ll 1$$

$$K_0(x) = -\ln(x) + \dots \quad , \quad K_m(x) = \frac{1}{2}(m-1)! \left(\frac{x}{2} \right)^{-m} + \dots \quad , \quad m \geq 1$$

$$I_m(x) \rightarrow \frac{e^x}{\sqrt{2\pi x}} \quad , \quad K_m(x) \rightarrow e^{-x} \sqrt{\frac{\pi}{2x}} \quad , \quad x \gg m+1$$

$$I_m(x) = i^{-m} J_m(ix) \quad , \quad K_m(x) = \frac{\pi}{2} i^{m+1} [J_m(ix) + i Y_m(ix)]$$

$$I_{-m}(x) = I_m(x) \quad , \quad K_{-m}(x) = K_m(x)$$

Spherical Bessel functions — $z_\ell(x)$ stands for either $j_\ell(x)$ or $n_\ell(x)$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - \frac{\ell(\ell+1)}{r^2} f = -k^2 f \implies f(r) = j_\ell(kr) \text{ or } f(r) = n_\ell(kr)$$

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \left(\frac{\sin x}{x} \right) \quad , \quad n_\ell(x) = -(-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \left(\frac{\cos x}{x} \right)$$

$$j_0(x) = \frac{\sin x}{x} \quad , \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad , \quad j_\ell(x) = \frac{x^\ell}{1 \cdot 3 \cdots (2\ell+1)} + \dots \quad x \ll 1$$

$$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad n_\ell(x) = -\frac{1 \cdot 3 \cdots (2\ell - 1)}{x^{\ell+1}} + \cdots \quad x \ll 1$$

$$\frac{2\ell + 1}{x} z_\ell(x) = z_{\ell-1}(x) + z_{\ell+1}(x), \quad (2\ell + 1) z'_\ell(x) = \ell z_{\ell-1}(x) - (\ell + 1) z_{\ell+1}(x)$$

Zeroes of $j_\ell(x)$ and $j'_\ell(x)$:

n	$j_\ell(x_{\ell,n}) = 0$				$j'_\ell(x'_{\ell,n}) = 0$		
	$x_{0,n}$	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x'_{0,n}$	$x'_{1,n}$	$x'_{2,n}$
1	3.1416	4.4934	5.7635	6.9880	0.0000	2.0816	3.3421
2	6.2832	7.7253	9.0950	10.4171	4.4934	5.9404	7.2900
3	9.4248	10.9041	12.3229	13.6980	7.7253	9.2059	10.6139
4	12.5664	14.0662	15.5146	16.9236	10.9041	12.4045	13.8461

Legendre polynomials

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_\ell}{dx} \right] = -\ell(\ell+1) P_\ell(x), \quad -1 < x < 1$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \quad P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

$$P_\ell(-x) = (-1)^\ell P_\ell(x), \quad P_\ell(1) = 1, \quad P_\ell(0) = (-1)^{\ell/2} \frac{1 \cdot 3 \cdots (\ell-1)}{2^{\ell/2} (\ell/2)!}, \quad \ell \text{ even}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell, \quad \int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell,\ell'}$$

$$P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell, \quad (\ell+1)P_{\ell+1} = (2\ell+1)xP_\ell - \ell P_{\ell-1}$$

$$P'_{\ell+1} = xP'_\ell + (\ell+1)P_\ell, \quad (x^2-1)P'_\ell = \ell[xP_\ell - P_{\ell-1}]$$

Associated Legendre functions

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_\ell^m}{dx} \right] - \frac{m^2}{1-x^2} P_\ell^m = -\ell(\ell+1) P_\ell^m(x), \quad |m| \leq \ell, \quad -1 < x < 1$$

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell, \quad P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell-m)!}{(\ell+m)!} \delta_{\ell,\ell'}$$

Spherical harmonics

$$\nabla^2 Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}$$

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_{\ell}^m(\cos\theta) e^{im\phi}, \quad Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$$

$$\int_0^{2\pi} \int_0^{\pi} Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) \sin\theta \, d\theta \, d\phi = \delta_{\ell, \ell'} \delta_{m, m'}$$

ℓ	0	m 1	2
0	$Y_{00} = 1/\sqrt{4\pi}$		
1	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$	
2	$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2}\right)$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi}$	$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi}$