

PS # 7

1. a. Begin w/ general sol'n.

$$V(x,t) = \sum_{n=1}^{\infty} A_n \rho_n(x) \cos(\omega_n t) \quad (1)$$

where $\rho_n(x) = \sin(n\pi \frac{x}{L})$, $\omega_n = \sqrt{\frac{T}{\sigma}} \cdot \frac{n\pi}{L}$

$$n = 1, 2, \dots$$

$\frac{\partial V}{\partial t} \Big|_{t=0} = 0$ automatically because (1) uses only $\cos(\omega t)$

$$V(x,0) = \sum_{n=1}^{\infty} A_n \rho_n(x) = \varepsilon \left| \sin(2\pi \frac{x}{L}) \right|$$

mult by $\rho_m(x)$ and $\int_0^L dx$

$$A_m \langle \rho_m | \rho_n \rangle = \varepsilon \int_0^L \sin(m\pi \frac{x}{L}) \left| \sin(2\pi \frac{x}{L}) \right| dx$$

Integral will vanish for even m.

odd m: $A_m = \frac{4\varepsilon}{L} \int_0^{L/2} \sin(m\pi \frac{x}{L}) \sin(2\pi \frac{x}{L}) dx$

$$A_m = \frac{4\varepsilon}{\pi} \int_0^{\pi/2} \sin(m\theta) \sin(2\theta) d\theta$$

using $\theta = \pi \frac{x}{L}$

b.) $U(x, t_0) = 0 \Rightarrow$ every term in series must vanish

$$\Rightarrow \cos(\omega_n t_0) = 0 \quad \text{for all odd } n$$

$$\omega_n t_0 = \frac{\pi}{2} M_n \quad \text{for some odd } M_n$$

$$= n \frac{\pi}{L} \sqrt{\frac{T}{\sigma}} t_0$$

$$\Rightarrow t_0 = \frac{L}{2} \sqrt{\frac{\sigma}{T}} \cdot \frac{M_n}{n} \quad \text{must be indep of } n$$

$$\Rightarrow \frac{M_n}{n} = \varphi \quad \text{or} \quad M_n = \varphi \cdot n \quad \text{both odd}$$

$$\Rightarrow \varphi = \text{odd integer.}$$

$$t_0 = \frac{L}{2} \sqrt{\frac{\sigma}{T}} \cdot \varphi, \quad \varphi = 1, 3, 5, \dots$$

c.) Normal mode n :

$$U_n(x, t) = A_n p_n(x) \cos(\omega_n t) = A_n \sinh\left(n\pi \frac{x}{L}\right) \cos(\omega_n t)$$

$$T_n = \int_0^L \frac{1}{2} \sigma \left(\frac{\partial U_n}{\partial t}\right)^2 dx = \frac{1}{2} \sigma A_n^2 \omega_n^2 \cos^2(\omega_n t) \underbrace{\int_0^L \sinh^2\left(n\pi \frac{x}{L}\right) dx}_{= L/2}$$

$$T_n = \frac{1}{4} \sigma \cdot L \omega_n^2 A_n^2 \cos^2(\omega_n t) = E_n \cos^2(\omega_n t)$$

↑
total energy of mode n .

$$= 2 \operatorname{sh}(m\theta) \operatorname{sh}(2\theta)$$

$$\begin{aligned}
 A_m &= \frac{2\varepsilon}{\pi} \int_0^{\pi/2} \left\{ \cos[(m-2)\theta] - \cos[(m+2)\theta] \right\} d\theta \\
 &= \frac{2\varepsilon}{\pi} \left\{ \frac{\operatorname{sh}[(m-2)\theta]}{m-2} - \frac{\operatorname{sh}[(m+2)\theta]}{m+2} \right\} \Big|_0^{\pi/2} \\
 &= \frac{2\varepsilon}{\pi} \left\{ \frac{\operatorname{sh}\left[\left(\frac{m}{2}-1\right)\pi\right]}{m-2} - \frac{\operatorname{sh}\left[\left(\frac{m}{2}+1\right)\pi\right]}{m+2} \right\} \\
 &= \frac{2\varepsilon}{\pi} (-1)^{\frac{m+1}{2}} \left[\frac{1}{m-2} - \frac{1}{m+2} \right]
 \end{aligned}$$

$$\boxed{A_m = \frac{8\varepsilon}{\pi} (-1)^{\frac{m+1}{2}} \frac{1}{m^2-4}} \quad m \text{ odd}$$

check: $U(x,0)$ is continuous but discontinuous in 1st deriv.

$$\Rightarrow A_m \sim \frac{1}{m^2} \quad \text{large } m \checkmark$$

$$A_1 = \frac{8\varepsilon}{3\pi} > 0 \quad \text{which agrees w/ } U(x,0) \geq 0$$

full solution:

$$\boxed{U(x,t) = \frac{8\varepsilon}{\pi} \sum_{\text{odd } n} \frac{(-1)^{\frac{n+1}{2}}}{n^2-4} \operatorname{sh}\left(n\pi \frac{x}{L}\right) \cos(\omega_n t)}$$

$$\boxed{\omega_n = n \frac{\pi}{L} \sqrt{\frac{T}{\sigma}}}$$

$$E_n = \frac{1}{4} \sigma L \omega_n^2 A_n^2 = \frac{\pi^2}{4} \frac{T}{L} n^2 A_n^2$$

$$E_n = 16 \frac{T}{L} \varepsilon^2 \frac{A^2}{n^2 - 4}$$

is energy in mode n (n is odd).

$$d.) V_0 = V(t_0) = \int_0^L \frac{1}{2} T \left(\frac{\partial V}{\partial x} \right)^2 dx$$

$$= \frac{\varepsilon^2}{2} T \left(\frac{2\pi}{L} \right)^2 \int_0^L \cos^2 \left(2\pi \frac{x}{L} \right) dx$$

$$V_0 = \pi^2 \frac{T}{L} \varepsilon^2 = E_{\text{tot}} = 4/2$$

$$\frac{E_1}{E_{\text{tot}}} = \frac{16}{9\pi^2} = \left(\frac{4}{3\pi} \right)^2 \approx 0.17 = 18\%$$

look @ given dist'n

$$\text{fraction } f_n = \frac{E_n}{E_{\text{tot}}} = \frac{16 n^2}{\pi^2 (n^2 - 4)^2} = \left[\frac{4n}{\pi(n^2 - 4)} \right]^2$$

$$f_1 = 0.18, f_3 = 0.58, f_5 = 0.09, f_7 = 0.04$$

most energy in mode $n=3$

85% energy in modes 1, 3 & 5

Fraction above $N \gg 1$: $F_N = \sum_{n=N}^{\infty} f_n$ remains

$$F_N = \left(\frac{4}{\pi} \right)^2 \sum_{n=N}^{\infty} \frac{1}{n^2} \approx \frac{8}{\pi^2 N} \Rightarrow F_9 \approx 0.09 \leftarrow$$

$$2. \quad \sigma(x) = \alpha x, \quad \tau(x) = \frac{1}{2} g \alpha x^2$$

a. Rayleigh-Ritz estimate

$$R\{f(x)\} = \frac{N\{f(x)\}}{D\{f(x)\}}$$

$$N\{f(x)\} = \int_0^L \tau(x) \left(\frac{\partial f}{\partial x}\right)^2 dx = \frac{1}{2} g \alpha \int_0^L x^2 dx = \frac{1}{6} g \alpha L^3$$

$$\begin{aligned} D\{f(x)\} &= \int_0^L \sigma(x) f^2(x) dx = \alpha \int_0^L x(L-x)^2 dx \\ &= \alpha L^4 \int_0^1 u(1-u)^2 du = \alpha L^4 \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\alpha L^4}{12} \end{aligned}$$

$$R\{f(x)\} = \frac{N}{D} = 2 \frac{g}{L} \geq \omega_1^2$$

$$\Rightarrow \boxed{\omega_1^2 \leq \sqrt{2} \sqrt{\frac{g}{L}}}$$

b.) $\rho(x) = A(x) \sin[\phi(x)]$

$$\frac{d}{dx} \left[\tau(x) \frac{d\rho}{dx} \right] \approx -\tau(x) \left| \frac{d\phi}{dx} \right|^2 \rho(x)$$

$$= -\omega^2 \sigma(x) \rho(x)$$

$$\Rightarrow \frac{d\phi}{dx} = \omega \sqrt{\frac{\sigma(x)}{\tau(x)}} = \omega \sqrt{\frac{2}{g}} \frac{1}{\sqrt{x}}$$

$$\varphi(L) = \int_0^L \frac{\partial \varphi}{\partial x} dx = \omega \sqrt{\frac{z}{g}} \int_0^L \frac{dx}{\sqrt{x}} = 2\sqrt{z} \omega \sqrt{\frac{L}{g}}$$

B.C. $\varphi(L) = 2\pi \Rightarrow$

$$\omega_n \approx n \frac{\pi}{2\sqrt{z}} \sqrt{\frac{g}{L}}$$

c.) propose $s = \sqrt{x}$

$$\frac{d}{dx} \Rightarrow \frac{ds}{dx} \cdot \frac{d}{ds} = \frac{1}{2} \frac{1}{s} \frac{d}{ds}$$

$$\frac{d}{dx} \left[\gamma(x) \frac{d\rho}{dx} \right] = \frac{\alpha g}{8} \frac{1}{s} \frac{d}{ds} \left[s^3 \frac{d\rho}{ds} \right]$$

$$= -\sigma(x) \omega^2 \rho = -\alpha s^2 \omega^2 \rho$$

$$\text{gives: } \left[\frac{d}{ds} \left[s^3 \frac{d\rho}{ds} \right] = -\frac{8}{g} \omega^2 \cdot s^3 \rho \right]$$

hint $\Rightarrow \rho(s) = \frac{1}{3} f(s)$

$$\frac{d\rho}{ds} = \frac{1}{3} f' - \frac{f(s)}{s^2}$$

$$\frac{d}{ds} \left[s^3 \frac{d\rho}{ds} \right] = \frac{d}{ds} \left[s^2 f' - s f \right] = s^2 f'' + s f' - f$$

equation for $f(s)$

$$s^2 f'' + s f' - f = -\frac{8}{g} \omega^2 s^2 f$$

or

$$f'' + \frac{1}{s} f' - \frac{1}{s^2} f = -\frac{8\omega^2}{g} f$$

solutions:

$$f(s) = J_1\left(\omega\sqrt{\frac{8}{g}} \cdot s\right)$$

or

$$f(s) = Y_1\left(\omega\sqrt{\frac{8}{g}} s\right) \leftarrow \text{not regular at } s=0$$

$$\Rightarrow p(x) = \frac{1}{\sqrt{x}} J_1\left(\omega\sqrt{\frac{8}{g}} \sqrt{x}\right)$$

satisfy B.C. @ $x=L$

$$p(L) = \frac{1}{\sqrt{L}} J_1\left(\omega\sqrt{\frac{8L}{g}}\right) = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{8L}} \cdot j_{1,n} \quad n = 0, 1, 2, \dots$$

d.) Fundamental β s

$$W_1 = \sqrt{\frac{g}{L}} \frac{j_{1,0}}{2.57} = \sqrt{\frac{g}{L}} \frac{3.832}{2.828} \approx 1.355 \sqrt{\frac{g}{L}}$$

upper bound : $W_1 \leq 1.414 \sqrt{\frac{g}{L}}$

β high by 4.4% excellent

asymptotics : $j_{1,n} \approx \pi n$

$$\Rightarrow W_n \approx \frac{\pi}{2.57} \sqrt{\frac{g}{L}} \cdot n$$

as predicted.

$n = 20$: $j_{1,20} \approx 63.611$

$$\Rightarrow W_{20} = 22.5 \sqrt{\frac{g}{L}}$$

compare to $\frac{\pi \cdot 20}{\sqrt{8}} \sqrt{\frac{g}{L}} \approx 22.2 \sqrt{\frac{g}{L}}$

off by 1.3% most excellent

e.) general $\rho_n(x) = \frac{1}{\sqrt{x}} J_1(j_{1,n} \sqrt{\frac{x}{L}})$

inner product $\langle \rho_m | \rho_n \rangle = \int_0^L \rho_m(x) \rho_n(x) \sigma(x) dx$

$$\langle \rho_m | \rho_n \rangle = \alpha \int_0^L J_1(j_{1,m} \sqrt{\frac{x}{L}}) J_1(j_{1,n} \sqrt{\frac{x}{L}}) dx$$

introduce : $z = \sqrt{\frac{x}{L}} \Rightarrow x = L z^2$

$$\Rightarrow dx = 2L z dz$$

$$\langle \rho_m | \rho_n \rangle = 2L \alpha \int_0^1 J_1(j_{1,m} z) J_1(j_{1,n} z) z dz$$

= 0 unless $m = n$ by orthogonality of Bessel Functions.

f.) general solution

$$U(x, t) = \sum_{n=0}^{\infty} A_n \rho_n(x) \cos(\omega_n t)$$

$$U(x, 0) = \sum_{n=0}^{\infty} A_n \rho_n(x) = \varepsilon \left(1 - \frac{x}{L}\right)$$

orthogonality : $A_n \langle \rho_n | \rho_n \rangle = \varepsilon \int_0^L \left(1 - \frac{x}{L}\right) \rho_n(x) \sigma(x) dx$

$$A_n \langle p_n | p_n \rangle = \varepsilon \alpha \int_0^L (1 - \frac{x}{L}) J_1(j_{1,n} \sqrt{\frac{x}{L}}) \sqrt{x} dx$$

use $x = L z^2$, $dx = 2L z dz$

$$A_n \langle p_n | p_n \rangle = 2 \varepsilon \alpha L^{3/2} \int_0^1 (1-z) J_1(j_{1,n} z) z^2 dz$$

$$A_n = \frac{2 \varepsilon \alpha L^{3/2}}{\langle p_n | p_n \rangle} \int_0^1 (1-z) J_1(j_{1,n} z) z^2 dz$$

F_n

$n = 0, 1, \dots$

$$\langle p_n | p_n \rangle = 2L \alpha \int_0^1 J_1^2(j_{1,n} z) z dz$$

M_n

$$A_n = \varepsilon \sqrt{L} \cdot \frac{F_n}{M_n}$$

$$A_n p_n(x) = \varepsilon \sqrt{\frac{L}{x}} J_1(j_{1,n} \sqrt{\frac{x}{L}}) \cdot \frac{F_n}{M_n}$$

has right units also $J_1(z) \sim \frac{1}{2} z$, $z \ll 1$

$$A_n p_n(x) \approx \frac{\varepsilon}{2} j_{1,n} \cdot \frac{F_n}{M_n}, \quad x \ll L$$

g.) Initial potential energy:

$$V_0 = V(\rho) = \frac{1}{2} \int_0^L \gamma(x) \left(\frac{dV}{dx} \right)^2 dx$$
$$= \frac{\varepsilon^2}{2} \cdot \frac{g\alpha}{2} \int_0^L x^2 \left(\frac{1}{L} \right)^2 dx$$

$$V_0 = \frac{\varepsilon^2 g\alpha}{4} \cdot \frac{L}{3} = \frac{\varepsilon^2 g\alpha L}{12}$$

Total energy

Mode 1:

$$v_1(x,t) = A_1 p_1(x) \cos(\omega_1 t)$$

$$T_1 = \frac{1}{2} \int_0^L \sigma(x) \left(\frac{\partial v_1}{\partial t} \right)^2 dx = \frac{1}{2} A_1^2 \omega_1^2 \sin^2(\omega_1 t) \langle p_1 | p_1 \rangle$$

$$E_1 = \frac{1}{2} A_1^2 \omega_1^2 \langle p_1 | p_1 \rangle$$

$$\text{v.l.e. } \langle p_n | p_n \rangle = 2L\alpha M_n, \quad A_n = \varepsilon \sqrt{L} \frac{F_n}{M_n}$$

$$\Rightarrow E_1 = \varepsilon^2 L^2 \alpha \omega_1^2 \frac{F_1^2}{M_1}$$

$$E_1 = \frac{\varepsilon^2 g\alpha L}{8} \cdot \frac{F_1^2}{M_1}$$