Hanging string

Uniform string of length L and mass $M = \sigma L$, fastened to ceiling at $x = 0$. Tension: $\tau(x) = \sigma g(L - x)$. Normal modes $\rho_n(x)$ satisfy

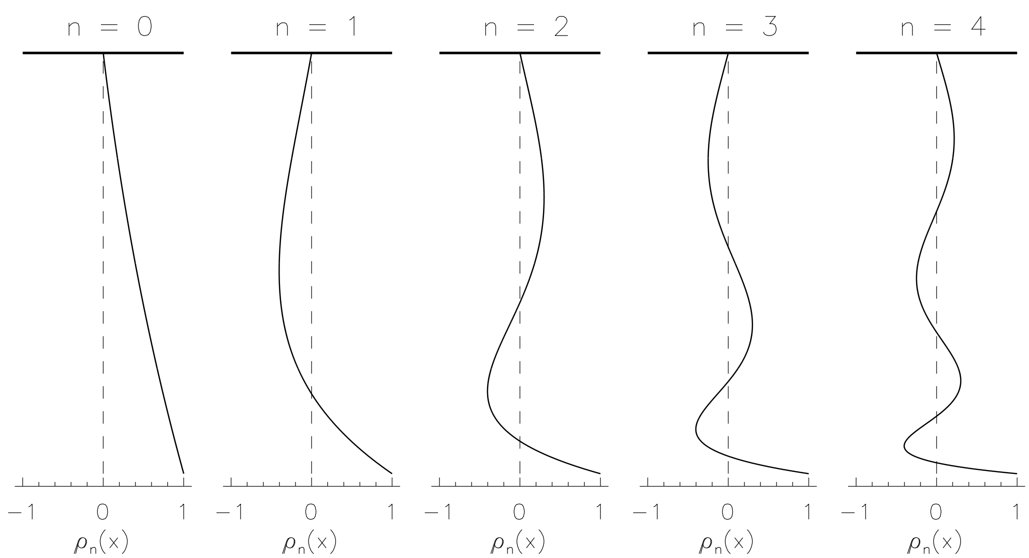
$$\frac{d}{dx}\Big[\Big(1 - \frac{x}{L}\Big)\frac{d\rho_n}{dx}\Big] \;=\; -\,\frac{\omega_n^2}{gL}\,\rho_n(x) \quad , \tag{1}$$

with boundary conditons

$$\rho_n(0) \;=\; 0 \quad , \quad \rho_n(x) \;\text{ regular as }\; x \to L \quad . \tag{2}$$

Solutions

$$\rho_n(x) \;=\; J_0\Bigg(j_{0,n}\sqrt{1 - \frac{x}{L}}\Bigg) \quad , \quad \omega_n \;=\; \tfrac{1}{2} j_{0,n}\sqrt{\frac{g}{L}} \quad , \quad n = 0,\, 1,\, 2,\, \ldots \tag{3}$$



Behavior near tip of string, $(L - x) \ll L$,

$$J_0(z) \;=\; 1 - \frac{z^2}{4} + \cdots \quad \Longrightarrow \quad \rho_n(x) \;\simeq\; 1 \;-\; \frac{j_{0,n}^2}{4}\Big(1 \;-\; \frac{x}{L}\Big) \;+\; \cdots \quad . \tag{4}$$

Orthogonality (assured as a solution of Sturm-Liouville eq.)

$$\begin{aligned}\langle \rho_m \,|\, \rho_n \rangle \;&=\; \int_0^L \rho_m(x)\,\rho_n(x)\,\sigma\,dx \;=\; \sigma \int_0^L J_0\Bigg(j_{0,m}\sqrt{1 - \frac{x}{L}}\Bigg) J_0\Bigg(j_{0,n}\sqrt{1 - \frac{x}{L}}\Bigg)\,dx \\ &=\; 2\sigma L \int_0^1 J_0(j_{0,m}u)\,J_0(j_{0,n}u)\,u\,du \;=\; M\,J_1^2(j_{0,n})\,\delta_{m,n} \quad . \end{aligned} \tag{5}$$

Solution to string equation w/ stationary initial condition

$$y(x,0) \;=\; f(x) \quad , \quad \left.\frac{\partial y}{\partial t}\right|_{t=0} \;=\; 0 \quad , \tag{6}$$

$$y(x,t) \;=\; \sum_{n=0}^{\infty} A_n \rho_n(x) \cos(\,\omega_n\, t\,) \quad , \quad A_n \;=\; \frac{\langle \rho_n \,|\, f\,\rangle}{\langle \rho_n \,|\, \rho_n\,\rangle} \tag{7}$$

Energies

$$V\{y(x,t)\} \;=\; \tfrac{1}{2}\int_0^L \tau(x) \left(\frac{\partial y}{\partial x}\right)^2 dx \;=\; \sum_{n=0}^{\infty} \tfrac{1}{2}\omega_n^2\, A_n^2\, \langle \rho_n \,|\, \rho_n\,\rangle\, \cos^2(\,\omega_n\, t\,) \tag{8}$$

$$T\{y(x,t)\} \;=\; \tfrac{1}{2}\int_0^L \sigma \left(\frac{\partial y}{\partial t}\right)^2 dx \;=\; \sum_{n=0}^{\infty} \tfrac{1}{2}\omega_n^2\, A_n^2\, \langle \rho_n \,|\, \rho_n\,\rangle\, \sin^2(\,\omega_n\, t\,) \tag{9}$$

$$H_n \;=\; \tfrac{1}{2}\omega_n^2\, A_n^2\, \langle \rho_n \,|\, \rho_n\,\rangle \;=\; F_n\, V_0 \quad , \quad V_0 \;=\; \tfrac{1}{2}Mg\int_0^L \left(1-\frac{x}{L}\right) f'^2(x)\, dx \tag{10}$$

For particular choice of initial state

$$f(x) \;=\; a\left(\frac{x}{L}\right)^p \quad \Longrightarrow \quad V_0 \;=\; \sigma g a^2\, \frac{p^2}{4p(2p-1)} \quad . \tag{11}$$

n	$j_{0,n}$	$\langle\rho_n\vert\rho_n\rangle/M$ $= J_1^2(j_{0,n})$	$f(x)=x/L$ A_n	$f(x)=x/L$ F_n	$f(x)=(x/L)^5$ A_n	$f(x)=(x/L)^5$ F_n
0	2.40483	0.26951	1.10802	0.95679	0.50162	0.35297
1	5.52008	0.11578	-0.13978	0.03446	0.44344	0.62436
2	8.65373	0.07369	0.04548	0.00571	0.06562	0.02138
3	11.79153	0.05404	-0.02099	0.00166	-0.01321	0.00118
4	14.93092	0.04266	0.01164	0.00064	0.00329	0.00009
5	18.07106	0.03524	-0.00722	0.00030	-0.00102	0.00001
6	21.21164	0.03002	0.00484	0.00016	0.00037	0.00000
7	24.35247	0.02615	-0.00343	0.00009	-0.00016	0.00000

Modes $n=0$ and $n=1$ are not commensurate

$$\frac{j_{0,1}}{j_{0,0}} \;=\; 2.29541727\ldots \quad \text{is irrational}$$

but

$$\frac{7}{3} \;=\; 2.3333 \quad , \quad \frac{179}{78} \;=\; 2.2949 \quad , \quad \frac{328185}{142974} \;=\; 2.29541735$$