

**Notes on symmetry**  
DANA LONGCOPE 9/18/2018

Physicists make frequent use of symmetry, when it appears in a problem. That term is used in a very precise way

A **symmetry** of a given system is a *transformation* which leaves the system's dynamics *unchanged*.

You probably already make some use of your intuitive understanding of symmetry — you “exploit” a problem's symmetry in some ways. Here I will show how to exploit the symmetry even more fully.

Some systems have more than one symmetry, meaning multiple ways to transform and leave dynamics unchanged. This set of transformations forms a mathematical object called a *group*, and many physicists study group theory in order to make the best possible use of symmetry. Here we have a far more modest objective: to exploit *one single* symmetry of a system.

## 1 Transformation of generalized coordinates

We start with a set of  $n$  generalized coordinates,  $(q_1, q_2, \dots, q_n) = \mathbf{q}$  and a Lagrangian  $L(\mathbf{q}, \dot{\mathbf{q}})$ . Now suppose there is some transformation which takes these coordinates to a new set,  $(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n) = \tilde{\mathbf{q}}$ . The transformation can be specified mathematically using  $n$  functions

$$\tilde{q}_i = F_i(\mathbf{q}) = F_i(q_1, q_2, \dots, q_n) \quad , \quad i = 1, 2, \dots, n \quad . \quad (1)$$

We see below that we almost never resort to such cumbersome formalism, but it helps keep the abstract concepts clear at first. There is also an inverse transformation

$$q_i = \tilde{F}_i(\tilde{\mathbf{q}}) = \tilde{F}_i(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n) \quad . \quad (2)$$

Time-differentiating the either transformation, and using the chain rule, gives a linear relation between the generalized velocities

$$\dot{\tilde{q}}_i = \sum_{j=1}^n \frac{\partial F_i}{\partial q_j} \dot{q}_j \quad , \quad \dot{q}_i = \sum_{j=1}^n \frac{\partial \tilde{F}_i}{\partial \tilde{q}_j} \dot{\tilde{q}}_j \quad . \quad (3)$$

While the velocities are linearly related, the matrices  $\partial F_i / \partial q_j$ , may depend on position, perhaps non-linearly.

Using the inverse transformations in the Lagrangian  $L(\mathbf{q}, \dot{\mathbf{q}})$  yields a new function of the new coordinates and velocities:  $\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ ,

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) = L[\mathbf{q}(\tilde{\mathbf{q}}), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})] \quad . \quad (4)$$

To be very clear,  $\tilde{L}$  has the same value as  $L$ , at corresponding points, but it is different function in the way it depends on its arguments. To obtain the equations of motion in the new coordinates one differentiates the new Lagrangian  $\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ .

To illustrate this very abstract discussion let's consider the simplest possible system: 2 free particles in 1 dimension. These have masses  $m_a$  and  $m_b$ , and we use generalized coordinates  $x_1$  and  $x_2$  for the two. The Lagrangian of this system is

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_a \dot{x}_1^2 + \frac{1}{2}m_b \dot{x}_2^2 \quad . \quad (5)$$

Consider the transformation which exchanges the particle *labels*:

$$\tilde{x}_1 = x_2 \quad , \quad \tilde{x}_2 = x_1 \quad . \quad (6)$$

The Lagrangian in these new coordinates is

$$\tilde{L}(\tilde{x}_1, \tilde{x}_2, \dot{\tilde{x}}_1, \dot{\tilde{x}}_2) = \frac{1}{2}m_a \dot{\tilde{x}}_2^2 + \frac{1}{2}m_b \dot{\tilde{x}}_1^2 \quad . \quad (7)$$

Symmetry transformations are almost always this simple — even obvious.

## 2 Symmetries

We now take the process one step further by **treating the new coordinates as if they were the old ones**. Formally this means dropping the  $\tilde{\phantom{x}}$  from the coordinates and velocities. What remains of the transformation is a new Lagrangian,  $\tilde{L}(\mathbf{q}, \dot{\mathbf{q}})$ , which may differ from the original one. In the 2-free particle example, the  $x_1 \leftrightarrow x_2$  swap produced the new Lagrangian

$$\tilde{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_b \dot{x}_1^2 + \frac{1}{2}m_a \dot{x}_2^2 \quad , \quad (8)$$

clearly differing from the one in eq. (5). Our transformation has had the effect of assigning to particle 1 mass  $m_b$  instead of  $m_a$ .

We are now in a position to give a more practical definition of symmetry.<sup>1</sup>

**A symmetry is a transformation of generalized coordinates that leaves the Lagrangian unchanged.**

By this we simply mean

$$\tilde{L}(\mathbf{q}, \dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}}) \quad . \quad (9)$$

The exchange transformation,  $x_1 \leftrightarrow x_2$ , in our simple example is a symmetry only if  $m_a = m_b$ . In that case the two particles are identical, so changing their labels will *not* change the Lagrangian. We then say that the system is symmetry under exchange of particles — but we really mean particle *labels*.

It is possible to show that, at least under conditions we demand for normal mode analysis, requiring the Lagrangian to remain unchanged requires that the kinetic energy and potential energy are each separately unchanged:

$$\tilde{T}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) \quad , \quad \tilde{V}(\mathbf{q}) = V(\mathbf{q}) \quad . \quad (10)$$

If the 2-particle system described above included a potential,  $V(x_1, x_2)$ , the interchange transformation would only be a symmetry if it left that function unchanged:  $V(x_1, x_2) = V(x_2, x_1)$ .

While this discussion has employed some sophisticated mathematics, physicists do not use anything like that in practice. It is usually very clear when a system has a symmetry. So clear, that we entirely skip the discussion of functions like  $F_i$ , and simply remark that the system is symmetric under a particular transformation. For example to the exchange of particle labels  $x_1 \leftrightarrow x_2$ , in the previous example. In the case of identical masses ( $m_a = m_b$ ) that fact is immediately obvious to any Physicist. A simple glance at the Lagrangian would confirm the fact with no further math needed. That is always how we will work.

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<sup>1</sup>Clearly if the Lagrangian is the same, then the dynamics which follow from it will also be the same. But demanding that the Lagrangian remain unchanged is more restrictive than demanding that the dynamics remain unchanged. It is possible for different Lagrangians to produce the same dynamics. For example if they differ by a constant. But most such examples are contrived, and our new definition is far simpler.

### 3 Transformation of small perturbations

We now consider the normal modes of a system (i.e. Lagrangian) with a symmetry  $\mathbf{q} \rightarrow \tilde{\mathbf{q}}$ . Normal modes are defined for a particular equilibrium  $\mathbf{q}^{(0)}$  which satisfies the  $n$  equations

$$\left. \frac{\partial V}{\partial q_i} \right|_{\mathbf{q}^{(0)}} = 0 \quad , \quad i = 1, 2, \dots, n \quad . \quad (11)$$

The potential energy is unchanged under a symmetry,  $\tilde{V}(\mathbf{q}) = V(\mathbf{q})$ , so the transformed point,  $\tilde{\mathbf{q}}^{(0)}$ , will also be an equilibrium. But it need not be *the same equilibrium*. When the two are the same,  $\tilde{\mathbf{q}}^{(0)} = \mathbf{q}^{(0)}$ , we say the *equilibrium is symmetric*. When this does not occur we have a case of *symmetry breaking*: a system with a particular symmetry has achieved an equilibrium which does not share that symmetry. As a simple example, consider including a potential  $V(x_1, x_2) = -\cos(x_1)\cos(x_2)$  in the system of identical particles ( $m_a = m_b$ ). This is obviously symmetric under the exchange  $x_1 \leftrightarrow x_2$ , so that remains a symmetry of this modified system. The equilibrium  $(x_1^{(0)}, x_2^{(0)}) = (0, 0)$  is symmetric under the exchange: it is a symmetric equilibrium. The equilibrium  $(x_1^{(0)}, x_2^{(0)}) = (\pi, -\pi)$  is not.

To obtain a normal mode we consider small perturbations to an equilibrium

$$\mathbf{q}(t) = \mathbf{q}^{(0)} + \boldsymbol{\eta}(t) \quad . \quad (12)$$

Small perturbation to the transformed coordinates are written similarly

$$\tilde{\mathbf{q}}(t) = \tilde{\mathbf{q}}^{(0)} + \tilde{\boldsymbol{\eta}}(t) \quad , \quad (13)$$

where  $\boldsymbol{\eta}(t)$  and  $\tilde{\boldsymbol{\eta}}(t)$  are linearly related through the derivative of the transformation

$$\tilde{\eta}_i(t) = \sum_{j=1}^n \left. \frac{\partial F_i}{\partial q_j} \right|_{\mathbf{q}^{(0)}} \eta_j(t) = \sum_{j=1}^n T_{ij} \eta_j(t) \quad . \quad (14)$$

Equation (3) shows that perturbation velocities are transformed by the same matrix

$$\dot{\tilde{\eta}}_i(t) = \sum_{j=1}^n T_{ij} \dot{\eta}_j(t) \quad . \quad (15)$$

Note that the matrix in eq. (3) was, in general, a function of coordinates — and possibly a non-linear function. We obtain the matrix  $\underline{\underline{T}}$  by evaluating that function at the equilibrium position  $\mathbf{q}^{(0)}$ . In many cases, such as our  $x_1 \leftrightarrow x_2$  exchange, the full transformation is linear, and thus defined by a constant matrix. In that case, the matrix transforming small perturbation,  $\underline{\underline{T}}$ , will be the same as the matrix defining the full transformation.

We see below that it is usually not necessary to compute the matrix  $\underline{\underline{T}}$ . For concreteness, however, the matrix for the exchange symmetry,  $x_1 \leftrightarrow x_2$  is

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad . \quad (16)$$

Potential and kinetic energies are quadratic forms in perturbations and their velocities

$$V = \frac{1}{2} \boldsymbol{\eta}^T \cdot \underline{\underline{V}} \cdot \boldsymbol{\eta} \quad , \quad T = \frac{1}{2} \dot{\boldsymbol{\eta}}^T \cdot \underline{\underline{M}} \cdot \dot{\boldsymbol{\eta}} \quad , \quad (17)$$

where matrices  $\underline{\underline{V}}$  and  $\underline{\underline{M}}$  are defined by the second derivatives

$$V_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\mathbf{q}^{(0)}} , \quad M_{ij} = \left. \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \right|_{\mathbf{q}^{(0)}} . \quad (18)$$

Equations of motion for the small perturbation

$$\underline{\underline{M}} \cdot \ddot{\boldsymbol{\eta}} = -\underline{\underline{V}} \cdot \boldsymbol{\eta} , \quad (19)$$

follow from the Lagrangian,  $L = T - V$ .

The potential and kinetic energies must have the same values in transformed coordinates

$$V = \frac{1}{2} \tilde{\boldsymbol{\eta}}^T \cdot \tilde{\underline{\underline{V}}} \cdot \tilde{\boldsymbol{\eta}} , \quad T = \frac{1}{2} \dot{\tilde{\boldsymbol{\eta}}}^T \cdot \tilde{\underline{\underline{M}}} \cdot \dot{\tilde{\boldsymbol{\eta}}} , \quad (20)$$

where the transformed matrices are

$$\tilde{V}_{ij} = \left. \frac{\partial^2 \tilde{V}}{\partial \tilde{q}_i \partial \tilde{q}_j} \right|_{\tilde{\mathbf{q}}^{(0)}} , \quad \tilde{M}_{ij} = \left. \frac{\partial^2 \tilde{T}}{\partial \dot{\tilde{q}}_i \partial \dot{\tilde{q}}_j} \right|_{\tilde{\mathbf{q}}^{(0)}} . \quad (21)$$

Equating the expressions for  $V$  in (17) and (20), and using eq. (14) to eliminate  $\tilde{\boldsymbol{\eta}}$ , shows that the potential matrix transforms in the manner standard for matrices:

$$\underline{\underline{V}} = \underline{\underline{T}}^T \cdot \tilde{\underline{\underline{V}}} \cdot \underline{\underline{T}} . \quad (22)$$

Following similar steps we find the mass matrix transforms the same way

$$\underline{\underline{M}} = \underline{\underline{T}}^T \cdot \tilde{\underline{\underline{M}}} \cdot \underline{\underline{T}} . \quad (23)$$

So a transformation of coordinates creates a linear transformation of small perturbations:  $\tilde{\boldsymbol{\eta}} = \underline{\underline{T}} \cdot \boldsymbol{\eta}$ . The mass matrix and potential matrix are also transformed by  $\underline{\underline{T}}$ . This is true of any transformation, whether it is a symmetry or not.

If the transformation  $\underline{\underline{T}}$  is a symmetry, and if we linearized about a symmetric equilibrium  $\mathbf{q}^{(0)}$ , then eq. (21) shows that  $\tilde{\underline{\underline{V}}} = \underline{\underline{V}}$  and  $\tilde{\underline{\underline{M}}} = \underline{\underline{M}}$ . This means these matrices satisfy the relations

$$\underline{\underline{V}} = \underline{\underline{T}}^T \cdot \underline{\underline{V}} \cdot \underline{\underline{T}} , \quad \underline{\underline{M}} = \underline{\underline{T}}^T \cdot \underline{\underline{M}} \cdot \underline{\underline{T}} . \quad (24)$$

These expressions provide very restrictive conditions which  $\underline{\underline{V}}$  and  $\underline{\underline{M}}$  will necessarily satisfy as a result of the system's symmetry. It is often more evident what these conditions are than the equations would suggest. For example, in the case of an exchange symmetry,  $x_1 \leftrightarrow x_2$ , the two diagonal elements must be the same,  $V_{11} = V_{22}$  and  $M_{11} = M_{22}$ . This is fairly obvious, but one can check that only a matrix of the form

$$\underline{\underline{V}} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} , \quad (25)$$

will satisfy eq. (24) when  $\underline{\underline{T}}$  is given by eq. (16).

## 4 Symmetries of normal modes

We now consider the normal modes of the symmetric system about its symmetric equilibrium. A normal mode is a perturbation

$$\boldsymbol{\eta}(t) = \boldsymbol{\rho} e^{-i\omega t} , \quad (26)$$

where  $\omega$  is its eigenfrequency. Placing this into eq. (19) gives the equation

$$\left( \omega^2 \underline{\underline{M}} - \underline{\underline{V}} \right) \cdot \boldsymbol{\rho} = 0 , \quad (27)$$

which a normal mode and its eigenfrequency must satisfy together. Replacing  $\underline{\underline{M}}$  and  $\underline{\underline{V}}$  using eq. (24) gives a relation

$$\underline{\underline{T}}^T \cdot \left( \omega^2 \underline{\underline{M}} - \underline{\underline{V}} \right) \cdot \underline{\underline{T}} \cdot \boldsymbol{\rho} = 0 . \quad (28)$$

Since the transformation has an inverse, so does the matrix  $\underline{\underline{T}}$ , and its transpose,  $\underline{\underline{T}}^T$ . Thus

$$\left( \omega^2 \underline{\underline{M}} - \underline{\underline{V}} \right) \cdot \underbrace{\left( \underline{\underline{T}} \cdot \boldsymbol{\rho} \right)}_{\tilde{\boldsymbol{\rho}}} = 0 . \quad (29)$$

Comparing eqs. (27) and (29) shows that both  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}} = \underline{\underline{T}} \cdot \boldsymbol{\rho}$  are normal modes **with the same eigenfrequency**. This is not entirely surprising since the transformation  $\boldsymbol{\rho} \rightarrow \tilde{\boldsymbol{\rho}}$  is a symmetry. It is, however, a statement with considerable power and consequence.

In the case that the eigenfrequency  $\omega$  is not degenerate, then  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}} = \underline{\underline{T}} \cdot \boldsymbol{\rho}$  must be **the same** normal mode — the only normal mode with that eigenfrequency. Since normal modes are only defined up to a multiplicative constant, however, the two vectors are not necessarily equal to one another. It is sufficient for them to be proportional with some constant of proportionality  $\lambda$

$$\tilde{\boldsymbol{\rho}} = \underline{\underline{T}} \cdot \boldsymbol{\rho} = \lambda \boldsymbol{\rho} . \quad (30)$$

This equation says that  $\boldsymbol{\rho}$  is an **eigenvector** of the transformation matrix  $\underline{\underline{T}}$ , and  $\lambda$  is its eigenvalue.

In the case that  $\omega$  is degenerate, it is possible to combine the degenerate modes into a normal mode  $\boldsymbol{\rho}$  that satisfies eq. (30). We will not show this, but you have seen a similar situation before. We have found a version in classical mechanics of the commutator theorem

**Commutator theorem:** Perturbations  $\boldsymbol{\eta}(t)$  to a symmetric equilibrium in a symmetric system, transform with a matrix  $\underline{\underline{T}}$ . It is always possible to find **normal modes** of the system,  $\boldsymbol{\rho}$ , which are *simultaneously eigenvectors* of the matrix  $\underline{\underline{T}}$ .

We are very clear here to distinguish between a **normal mode**, which satisfies eq. (27), and an **eigenvector** of  $\underline{\underline{T}}$ , which satisfies eq. (30). The amazing thing is that  $\boldsymbol{\rho}$  can be both things at once (the possibility of degeneracy requires that we use the weasel words “It is always possible to find”.)

You have probably encountered a very similar situation in the commutator theorem of quantum mechanics. To see how these are related we must make use of the additional fact that the transformation matrix is orthogonal<sup>2</sup>  $\underline{\underline{T}}^T = \underline{\underline{T}}^{-1}$ . Using that, the symmetry condition eq. (24) can be rewritten

$$\underline{\underline{T}} \cdot \underline{\underline{V}} = \underline{\underline{V}} \cdot \underline{\underline{T}} , \quad (31)$$

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<sup>2</sup>We have not done the work to show this, because we could obtain our result without doing so.

meaning  $\underline{T}$  and  $\underline{V}$  **commute**:  $[\underline{T}, \underline{V}] = 0$ . The same applies to  $\underline{M}$ , so  $\underline{T}$  commutes with the matrix  $\omega^2 \underline{M} - \underline{V}$ . Naturally, the results we know from quantum mechanics apply to those two matrices. That result also follows from symmetry.

This is extremely useful because it is often far easier to identify eigenvectors of the symmetry transformation than to compute normal modes. In some cases we can avoid computing the normal modes altogether. At the very least we can check the normal modes we laboriously compute against the symmetry condition.

## 5 Continuous symmetries — conserved momenta

So far we have been very vague about which kinds of transformations we are talking about. There are basically two kinds: continuous symmetries and discrete symmetries. A continuous symmetry involves a transformation which is one from a continuous set of transformations. One important example is the translation of the system in one direction,  $q_1 \rightarrow q_1 + a$ , for any real number  $a$ . Since  $a$  can be any real number, there are a continuous set of translations. If the Lagrangian is symmetric to this particular transformation, then according to eq. (9), we must have

$$\tilde{L}(q_1, q_2, \dots, \dot{q}_n) = L(q_1 + a, q_2, \dots, \dot{q}_n) = L(q_1, q_2, \dots, \dot{q}_n) . \quad (32)$$

Taking the total derivative of this expression w.r.t.  $a$  gives

$$\frac{\partial L}{\partial q_1} = 0 , \quad (33)$$

after using the chain rule on the middle expression and noting that the rhs does not depend on  $a$ . Symmetry to translation implies a cyclic coordinate, which in turn implies that the conjugate momentum

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} , \quad (34)$$

is a constant of the motion.

In this example we must have  $\partial V / \partial q_1 = 0$ , and all second derivatives

$$\frac{\partial^2 V}{\partial q_1 \partial q_j} = V_{1,j} = V_{j,1} = 0 , \quad (35)$$

as well. Using in eq. (19) a generalized normal mode,  $\boldsymbol{\eta}(t) = \boldsymbol{\rho}^{(1)} f(t)$  with  $\rho_j^{(1)} = \delta_{1,j}$ , (the Kronecker delta), yields the equation

$$\ddot{f}(t) \underline{M} \cdot \boldsymbol{\rho}^{(1)} = \ddot{f}(t) \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ \vdots \\ M_{n,1} \end{bmatrix} = -f(t) \underline{V} \cdot \boldsymbol{\rho}^{(1)} = -f(t) \begin{bmatrix} V_{1,1} \\ V_{2,1} \\ \vdots \\ V_{n,1} \end{bmatrix} = 0 , \quad (36)$$

since the final column vector is  $V_{j,1} = 0$ , from above. If the first column vector were to entirely vanish,  $M_{j,1} = 0$ , there would have been no kinetic energy associated with the first degree of freedom. This would be a singular situation and we discount it. This leave the requirement  $\ddot{f} = 0$ , which means  $f = bt + c$  for arbitrary  $b$  and  $c$ . Had we proposed the usual normal mode form,  $f(t) = e^{-i\omega_1 t}$ , we would have  $\ddot{f} = -\omega_1^2 f(t) = 0$ , which would lead us to conclude that

$\omega_1 = 0$ . A continuous symmetry therefore demands a zero-frequency normal mode. The generic time dependence of such a normal mode is actually linear

$$\boldsymbol{\eta}(t) = (bt + c) \boldsymbol{\rho}^{(1)} \quad , \quad (37)$$

rather than exponential or strictly constant.

We can also dot eq. (19) on the left by the row vector,  $\boldsymbol{\rho}^{(1)T} = [1, 0, \dots, 0]$  to obtain

$$\boldsymbol{\rho}^{(1)T} \cdot \underline{\underline{M}} \cdot \ddot{\boldsymbol{\eta}} = \frac{d}{dt} \left\{ \boldsymbol{\rho}^{(1)T} \cdot \underline{\underline{M}} \cdot \dot{\boldsymbol{\eta}} \right\} = - \boldsymbol{\rho}^{(1)T} \cdot \underline{\underline{V}} = - \left[ V_{1,1}, V_{1,2}, \dots, V_{1,n} \right] = 0 \quad , \quad (38)$$

since the final row vector is  $V_{1,j} = 0$ , due to the continuous symmetry. The result is that the term in curly brackets,

$$\boldsymbol{\rho}^{(1)T} \cdot \underline{\underline{M}} \cdot \dot{\boldsymbol{\eta}} = \sum_{j=1}^n M_{1,j} \dot{\eta}_j = \left. \frac{\partial L}{\partial \dot{q}_1} \right|_{\mathbf{q}^{(0)}} = p_1 \quad , \quad (39)$$

is a constant of the motion. It is the conserved momentum  $p_1$ .

This shows that three things are inter-related: a continuous symmetry, a conserved momentum, a zero-frequency normal mode. It turns out this is not peculiar to this particular example – these three things **always** occur together, and any one implies the other two. This is known as *Noether's theorem*, but we will be better able to demonstrate this in complete generality once we introduce Hamiltonians and Poisson brackets.

## 6 Discrete symmetries of order $m$

A discrete symmetry involves a transformation that cannot be done continuously, or part way. The exchange  $x_1 \leftrightarrow x_2$  is a perfect example: we cannot swap labels just part way.

Any discrete transformation can be repeated to produce another transformation. First transform  $\mathbf{q} \rightarrow \tilde{\mathbf{q}}$ , and then  $\tilde{\mathbf{q}} \rightarrow \tilde{\tilde{\mathbf{q}}}$  to produce the full transformation  $\mathbf{q} \rightarrow \tilde{\tilde{\mathbf{q}}}$ . We can keep going by repeating and repeating. A transformation is said to be of **order**  $m$  if repeating it  $m$  times returns us to the original coordinates. For example, the exchange  $x_1 \leftrightarrow x_2$  is of order 2: exchange and then exchange again, and you are back to the original variables. There are many order-2 symmetries in Physics.

Small perturbations  $\boldsymbol{\eta}$  transform through the matrix  $\underline{\underline{T}}$ , and repeated transformation is achieved by matrix multiplication. Repeating it  $m$  times, i.e.  $\underline{\underline{T}}^m$ , returns us to the original coordinates, so

$$\underline{\underline{T}}^m = \underline{\underline{I}} \quad , \quad (40)$$

where  $\underline{\underline{I}}$  is the identity matrix.

Equation (40) tells us immediately what eigenvalues the matrix  $\underline{\underline{T}}$  can have. If  $\lambda$  is an eigenvalue of  $\underline{\underline{T}}$ , then  $\lambda^m$  is the corresponding eigenvalue of  $\underline{\underline{T}}^m$ . The only eigenvalue of the identity matrix,  $\underline{\underline{I}}$ , is one. Thus eq. (40) means the eigenvalues of an order- $m$  transformation  $\underline{\underline{T}}$  must satisfy the equation

$$\lambda^m = 1 \quad . \quad (41)$$

There are  $m$  different solutions to this equation, known as the  $m^{\text{th}}$  roots of unity

$$\lambda_s = \exp\left(2\pi i \frac{s}{m}\right) \quad , \quad s = 0, 1, \dots, m-1 \quad . \quad (42)$$

Depending on its size, and actual definition, the eigenvalues of the matrix  $\underline{T}$  will be some or all of these. Some of these may be degenerate eigenvalues.

What does this mean for the normal modes of a system with an order- $m$  symmetry? According to the commutator theorem, we can find normal modes of this system which are also eigenvectors of  $\underline{T}$ . Each of these eigenvectors will have one of the  $m$  eigenvalues given in eq. (42). The normal modes may thus be classified into groups according to this value.

### 6.1 Order-2 symmetry: symmetric and anti-symmetric modes

Order-2 symmetries are very common. Taking  $m = 2$  in eq. (41) shows that  $\lambda^2 = 1$ , meaning  $\lambda = \pm 1$ . This means that normal modes of a system with order-2 symmetry come in two varieties: modes unchanged by the transformation ( $\lambda = +1$ ), called *symmetric* modes; and modes that change sign under transformation ( $\lambda = -1$ ), called *anti-symmetric* modes.

It is sometimes possible to find the eigenvectors, and thereby the normal modes, based on this argument alone. Consider an eigenvector  $\mathbf{v}$  of the exchange operation  $x_1 \leftrightarrow x_2$ . This transforms according to the rule  $\tilde{v}_1 = v_2$ , which is the definition of the transformation. Since it is an eigenvector,

$$\tilde{\mathbf{v}} = \underline{T} \cdot \mathbf{v} = \lambda \mathbf{v} \quad , \quad (43)$$

whose top row reads  $\tilde{v}_1 = \lambda v_1$ . Combing these two facts gives the relation

$$v_2 = \lambda v_1 \quad . \quad (44)$$

Finally, the eigenvector  $\mathbf{v}$  is only defined up to a constant factor. We fix this by setting  $v_1 = 1$  and obtain the general eigenvector

$$\mathbf{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad . \quad (45)$$

The symmetric and anti-symmetric eigenvectors of the exchange operation

$$\mathbf{v}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}^{(-)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad , \quad (46)$$

come from taking  $\lambda = +1$  and  $\lambda = -1$  respectively. It is possible to arrive at these two eigenvectors directly from the matrix  $\underline{T}$  in eq. (16). We have found them with slightly less math.

It is also possible to reach the same answer with almost no math at all. The symmetric mode is a 2-element vector which does not change if its rows are swapped. The anti-symmetric mode is a 2-element vector which changes signs when its rows are swapped. The only answer to each condition, up to a irrelevant scaling factor, is  $\mathbf{v}^{(+)}$  and  $\mathbf{v}^{(-)}$ , shown in eq. (46). In this case each eigenvector is unique, and can be found by simply thinking about what condition it must satisfy under the transformation of row-swapping — equivalent to coordinate exchange.

This has told us something truly profound. If I have any 2-degree-of-freedom Lagrangian at all  $L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ , which is symmetric under exchange  $x_1 \leftrightarrow x_2$ , and I linearize it about a symmetric equilibrium, then its 2 normal modes must be the vectors given in eq. (46). There must be two, and they must be different. They might have the same eigenfrequency, in which case every vector is a normal mode, including those given in eq. (46). That is where the “it is always possible...” caveat comes in. Thus we have found the normal modes by symmetry alone — even though I never even told you what the Lagrangian was.



Lest you think that you have every answer you'll ever need, let's consider a slightly more complicated case: 3 particles in one dimension with coordinates  $(x_1, x_2, x_3)$ . The first two particles are identical, but the third is different. The system will therefore be symmetric to the same exchange,  $x_1 \leftrightarrow x_2$ , which is given explicitly by

$$\tilde{x}_1 = x_2 \quad , \quad \tilde{x}_2 = x_1 \quad , \quad \tilde{x}_3 = x_3 \quad .$$

Since particle 3 is unique the transformation cannot affect it:  $\tilde{x}_3 = x_3$ . This is an order-2 symmetry, so I can find normal modes which are either symmetric or anti-symmetric under the transformation.

Following the arguments given above, an eigenvector  $\mathbf{v}$  must have the following properties

$$v_2 = \lambda v_1 \quad , \quad v_3 = \lambda v_3 \quad .$$

The first relation matches, eq. (44), and the second comes from following the same steps beginning with  $\tilde{v}_3 = v_3$ . For the anti-symmetric mode the second relation,  $v_3 = -v_3$ , tells us that  $v_3 = 0$  and thus

$$\mathbf{v}^{(-)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} . \tag{47}$$

But for the symmetric mode that relation becomes a tautology:  $v_3 = v_3$ , which can be satisfied by any number at all. The eigenvalue  $\lambda = +1$  is thus degenerate, with eigenvectors

$$\mathbf{v}^{(+)} = \begin{bmatrix} a \\ a \\ b \end{bmatrix} , \tag{48}$$

for arbitrary choice of  $a$  and  $b$ .

The three normal modes of this system must be as follows. One will be the unique anti-symmetric mode given by eq. (47),  $\boldsymbol{\rho}^{(1)} = \mathbf{v}^{(-)}$ . The remaining two,  $\boldsymbol{\rho}^{(2)}$  and  $\boldsymbol{\rho}^{(3)}$ , will be symmetric modes each of the form (48), but with different values of  $a$  and  $b$ . In order to find these we must actually solve the normal mode equation (27). Yes, we still need to do honest work now and again. But this time we're only looking for 2 of the 3 normal modes — we have already found  $\boldsymbol{\rho}^{(1)}$ .

## 6.2 Periodic chain of identical masses

Now let's consider  $n$  identical elements or particles, arranged in a linear array, and numbered  $1, 2, \dots, n$ . We take each to have only a single degree of freedom, whose generalized coordinate is  $q_j$ . The full system thus has  $n$  degrees of freedom, whose coordinates are  $(q_1, q_2, \dots, q_n)$ . The elements will interact with one another through some potential,  $V(q_1, q_2, \dots, q_n)$ , about which we will say nothing more.

There is almost a symmetry here since all the element are identical. But if the chain has ends at  $j = 1$  and  $j = n$ , then those elements have only a single neighbor ( $j = 2$  and  $j = n - 1$  respectively) while all other have two. This spoils the symmetry of the problem. We must remedy this if we are to use symmetry to find the normal modes. We do so by making the chain **periodic**. Doing so gives  $j = 1$  two neighbors like all the rest:  $j = 2$ , to its right, and  $j = n$ , to its left. And  $j = n$  now has two neighbors as well. Periodic boundaries like this often seem artificial, so why do Physicists use them so often? The answer is now evident — in order to give the system a symmetry they can exploit.

Thanks to the periodic boundaries the system is symmetric to a cyclic shift of coordinate labels:

$$\tilde{q}_1 = q_2 \quad , \quad \tilde{q}_2 = q_3 \quad , \dots , \quad \tilde{q}_j = q_{j+1} \quad , \dots , \quad \tilde{q}_{n-1} = q_n \quad , \quad \tilde{q}_n = q_1 \quad . \quad (49)$$

This is a linear transformation, so the perturbations transform in the very same way

$$\tilde{\eta}_1 = \eta_2 \quad , \quad \tilde{\eta}_2 = \eta_3 \quad , \dots , \quad \tilde{\eta}_j = \eta_{j+1} \quad , \dots , \quad \tilde{\eta}_{n-1} = \eta_n \quad , \quad \tilde{\eta}_n = \eta_1 \quad . \quad (50)$$

These relations are all we need, so we never go to the trouble of writing  $\underline{\underline{T}}$  as a huge  $n \times n$  matrix.

It is evident that this symmetry is of order  $n$ : shifting a label  $n$  times returns it to its original place. The eigenvalues of the transformation matrix,  $\underline{\underline{T}}$ , are given by eq. (42)

$$\lambda_s = \exp\left(2\pi i \frac{s}{n}\right) \quad , \quad s = 0, 1, \dots, n-1 \quad . \quad (51)$$

To obtain the eigenvector,  $\mathbf{v}$ , we consider the transformation  $\tilde{v}_j = v_{j+1}$ , from eq. (50). The  $j^{\text{th}}$  row of the eigenvector equation, eq. (43), is  $\tilde{v}_j = \lambda_s v_j$ . Combining these expressions yields

$$v_{j+1} = \lambda_s v_j \quad , \quad (52)$$

which means

$$v_{j+1} = \lambda_s^j v_1 \quad . \quad (53)$$

In this case we fix the arbitrary scaling by setting  $v_1 = \lambda_s$ , to yield all  $n$  eigenvectors

$$v_j^{(s)} = \lambda_s^j = \exp\left(2\pi i \frac{js}{n}\right) \quad , \quad s = 0, 1, \dots, n-1 \quad . \quad (54)$$

These form a complete basis of  $n$  vectors, so every eigenvalue  $\lambda_s$ , has one and only one eigenvector,  $\mathbf{v}^{(s)}$ .

Without degeneracy of  $\lambda_s$ , we need not worry about the “it is always possible to find ...” caveat in the commutator theorem. We can thus say with certainty that every one of the eigenvectors of  $\underline{\underline{T}}$ , is a normal mode vector of the perturbed system:

$$\rho_j^{(s)} = \exp\left(2\pi i \frac{js}{n}\right) \quad , \quad s = 0, 1, \dots, n-1 \quad . \quad (55)$$

We have managed to find all the normal mode vectors of this system without ever writing down its Lagrangian! Indeed, eq. (55) gives the normal mode vectors for **every Lagrangian** which has the periodic chain symmetry: it is symmetric under a cyclic label shift. Some people introduce that expression as a kind of *ansatz*. But it is more than an *ansatz* — it is the answer. We still do not know the eigenfrequencies corresponding to the normal modes. To find those we will need to use the actual Lagrangian. But we see below that we can predict how many eigenfrequencies will be degenerate, and which of the normal mode vectors  $\rho^{(s)}$  will form degenerate pairs.

### 6.3 Degeneracies for symmetries of order $m \geq 3$

For order  $m \geq 3$  some of the eigenvalues given in eq. (42) are complex. For odd  $m$ , all but  $s = 0$  are complex, and for even  $m$ , all but  $s = 0$  and  $m/2$ . These form complex conjugate pairs:

$$\lambda_s^* = \lambda_{m-s} \quad . \quad (56)$$

All elements of the matrix  $\underline{T}$  must be real, so taking the complex conjugate of the eigenvector equation, eq. (43), shows that the eigenvectors are complex as well, with

$$\mathbf{v}^{(s)*} = \mathbf{v}^{(m-s)} . \quad (57)$$

This is true of the eigenvectors of the cyclic shift relabeling symmetry, given in eq. (54). Here we see it is true of every symmetry with order  $m \geq 3$ .

On the other hand, normal mode vectors,  $\boldsymbol{\rho}^{(s)}$ , need not be complex. It has been established elsewhere that the any eigenfrequency squared,  $\omega_s^2$ , is always real. The matrices  $\underline{M}$  and  $\underline{V}$  have only real elements, so taking the complex conjugate of eq. (27) shows that  $\boldsymbol{\rho}^{(s)}$  and  $\boldsymbol{\rho}^{(s)*}$  are normal modes with the same eigenfrequency. Either they are the same, and  $\boldsymbol{\rho}^{(s)}$  is real,<sup>3</sup> or they are degenerate. In the latter case, they can be combined to form 2 purely real normal mode vectors corresponding to the real and imaginary parts of  $\boldsymbol{\rho}^{(s)}$ .

The upshot is that any system with symmetry of order  $m \geq 3$  must have at least some degenerate eigenfrequencies. The transformation matrix will have complex eigenvalues,  $\lambda_s$ , with complex eigenvectors  $\mathbf{v}^{(s)}$ . The complex conjugates of these eigenvectors have distinct eigenvalues. The commutator theorem says it is possible to find normal mode vectors which match these, and are therefore also complex. But we argued above that the real and imaginary part of a complex normal mode vector must be separate normal modes from a degenerate eigenfrequency.

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<sup>3</sup>Actually, they may be proportional with a possibly complex constant of proportionality. In that case  $\boldsymbol{\rho}^{(s)}$  can be made real by multiplying by a related factor.