

Trigonometric functions:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	3π	4π	$n\pi$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	+1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	0	0	0
$\cos(x)$	+1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	+1	-1	+1	$(-1)^n$

$$\sin^2 x + \cos^2 x = 1 \quad , \quad 1 + \tan^2 x = \sec^2 x \quad , \quad 1 + \cot^2 x = \csc^2 x$$

$$e^{ix} = \cos x + i \sin x \quad , \quad \cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad , \quad \sin(x) = -i\frac{1}{2}(e^{ix} - e^{-ix})$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad , \quad \sin(2a) = 2 \sin a \cos a$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1$$

Hyperbolic functions:

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad , \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \quad , \quad e^x = \cosh(x) + \sinh(x)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad , \quad 1 - \tanh^2 x = \operatorname{sech}^2 x \quad ,$$

$$\cos(ix) = \cosh(x) \quad , \quad \sin(ix) = i \sinh(x) \quad , \quad \tan(ix) = i \tanh(x)$$

Indefinite Integrals (anti-derivatives):

$$\int \sin(x) \cos(x) dx = -\frac{1}{4} \cos(2x)$$

$$\int \cos^2(x) dx = \frac{1}{2}x + \frac{1}{4} \sin(2x) \quad , \quad \int \sin^2(x) dx = \frac{1}{2}x - \frac{1}{4} \sin(2x)$$

$$\int x \cos(ax) dx = \frac{1}{a^2} \cos(ax) + \frac{x}{a} \sin(ax) \quad , \quad \int x \sin(ax) dx = \frac{1}{a^2} \sin(ax) - \frac{x}{a} \cos(ax)$$

$$\int x^2 \cos(ax) dx = \frac{2x}{a^2} \cos(ax) + \frac{a^2 x^2 - 2}{a^3} \sin(ax) \quad ,$$

$$\int x^2 \sin(ax) dx = \frac{2x}{a^2} \sin(ax) - \frac{a^2 x^2 - 2}{a^3} \cos(ax) \quad ,$$

$$\int x^\nu J_{\nu-1}(x) dx = x^\nu J_\nu(x) \quad , \quad \int x^\nu I_{\nu-1}(x) dx = x^\nu I_\nu(x)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) \quad , \quad \int \frac{dx}{1+x^2} = \tan^{-1}(x) \quad ,$$

$$\int \frac{x dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} \quad , \quad \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$$

$$\int \frac{x dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2+a^2}} \quad , \quad \int \frac{x dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

Vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \quad , \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad ,$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times (\nabla f) = 0 \quad , \quad \nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad , \quad \nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (f\mathbf{a}) = \nabla f \cdot \mathbf{a} + f\nabla \cdot \mathbf{a} \quad , \quad \nabla \times (f\mathbf{a}) = \nabla f \times \mathbf{a} + f\nabla \times \mathbf{a}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

Cylindrical Coordinates: (r, ϕ, z) , $x = r \cos \phi$, $y = r \sin \phi$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \quad , \quad d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}} \quad , \quad dV = r dr d\phi dz \quad ,$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = \left[\frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{r}} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{\boldsymbol{\phi}} + \left[\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi} \right] \frac{\hat{\mathbf{z}}}{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \quad , \quad \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates: (r, θ, ϕ) , $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}} \quad , \quad dV = r^2 \sin \theta dr d\theta d\phi \quad ,$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\nabla \times \mathbf{v} = \left[\frac{\partial(\sin \theta v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right] \frac{\hat{\mathbf{r}}}{r \sin \theta} + \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial(rv_\phi)}{\partial r} \right] \frac{\hat{\boldsymbol{\theta}}}{r} + \left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] \frac{\hat{\boldsymbol{\phi}}}{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Legendre polynomials:

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_\ell}{dx} \right] = -\ell(\ell+1) P_\ell(x) \quad , \quad -1 < x < 1$$

$$\begin{aligned} P_0(x) &= 1 & P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} & P_4(x) &= \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} \\ P_1(x) &= x & P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x & P_5(x) &= \frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x \end{aligned}$$

$$P_\ell(-x) = (-1)^\ell P_\ell(x) \quad , \quad P_\ell(1) = 1 \quad , \quad P_\ell(0) = (-1)^{\ell/2} \frac{(\ell-1)!!}{\ell!!} \quad , \quad \ell \text{ even}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d}{dx} (x^2-1)^\ell \quad , \quad \int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell,\ell'}$$

$$P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell \quad , \quad (\ell+1)P_{\ell+1} = (2\ell+1)xP_\ell - \ell P_{\ell-1}$$

$$P'_{\ell+1} = xP'_\ell + (\ell+1)P_\ell \quad , \quad (x^2-1)P'_\ell = \ell[xP_\ell - P_{\ell-1}]$$

Associated Legendre functions:

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_\ell^m}{dx} \right] - \frac{m^2}{1-x^2} P_\ell^m = -\ell(\ell+1) P_\ell^m(x) \quad , \quad |m| \leq \ell$$

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell^m \quad , \quad P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell-m)!}{(\ell+m)!} \delta_{\ell,\ell'}$$

Spherical harmonics:

$$\nabla^2 Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}$$

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi} \quad , \quad Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$$

$$\int_0^{2\pi} \int_0^\pi Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) \sin \theta d\theta d\phi = \delta_{\ell,\ell'} \delta_{m,m'}$$

ℓ	0	m 1	2
0	$Y_{00} = 1/\sqrt{4\pi}$		
1	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$	
2	$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$	$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}$

Bessel functions: — $\Omega_m(x)$ stands for either $J_m(x)$ or $Y_m(x)$

$$\frac{1}{x} \frac{d}{dx} \left[x \frac{dy}{dx} \right] - \frac{m^2}{x^2} y = -k^2 y \quad \Longrightarrow \quad y(x) = J_m(kx) \quad \text{or} \quad y(x) = Y_m(kx)$$

$$J_{-m}(x) = (-1)^m J_m(x) \quad , \quad m \text{ integer}$$

$$\Omega_{m-1}(x) + \Omega_{m+1}(x) = \frac{2m}{x} \Omega_m(x) \quad , \quad \Omega_{m-1}(x) - \Omega_{m+1}(x) = 2\Omega'_m(x)$$

$$\Omega'_m(x) = \Omega_{m-1}(x) - \frac{m}{x} \Omega_m(x) \quad , \quad \Omega'_m(x) = -\Omega_{m+1}(x) + \frac{m}{x} \Omega_m(x)$$

$$J_m(x) = \frac{1}{m!} \left(\frac{x}{2} \right)^m + \dots \quad , \quad x \ll 1$$

$$Y_0(x) = \frac{2}{\pi} \ln(x) + \dots \quad , \quad Y_m(x) = -\frac{(m-1)!}{\pi} \left(\frac{x}{2} \right)^{-m} + \dots \quad , \quad m \geq 1$$

$$J_m(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos \left[x - \left(m + \frac{1}{2} \right) \frac{\pi}{2} \right] \quad , \quad x \gg m + 1$$

$$Y_m(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin \left[x - \left(m + \frac{1}{2} \right) \frac{\pi}{2} \right] \quad , \quad x \gg m + 1$$

$$\int_0^1 J_m(j_{m,n}x) J_m(j_{m,n'}x) x dx = \frac{1}{2} J_{m+1}^2(j_{m,n}) \delta_{n,n'} = \frac{1}{2} \left[J'_m(j_{m,n}) \right]^2 \delta_{n,n'} \quad , \quad J_m(j_{m,n}) = 0$$

n	$j_{0,n}$	$J_1(j_{0,n})$	$j_{1,n}$	$J_2(j_{1,n})$	$j_{2,n}$	$J_3(j_{2,n})$	$j_{3,n}$	$J_4(j_{3,n})$	$j_{4,n}$	$J_5(j_{4,n})$
0	2.40	0.519	3.83	0.403	5.14	0.340	6.38	0.298	7.59	0.268
1	5.52	-0.340	7.02	-0.300	8.42	-0.271	9.76	-0.249	11.06	-0.232
2	8.65	0.271	10.17	0.250	11.62	0.232	13.02	0.218	14.37	0.206
3	11.79	-0.232	13.32	-0.218	14.80	-0.207	16.22	-0.196	17.62	-0.188
4	14.93	0.207	16.47	0.196	17.96	0.188	19.41	0.180	20.83	0.173
5	18.07	-0.188	19.62	-0.180	21.12	-0.173	22.58	-0.167	24.02	-0.162

Modified Bessel functions:

$$\frac{1}{x} \frac{d}{dx} \left[x \frac{dy}{dx} \right] - \frac{m^2}{x^2} y = +k^2 y \quad \Longrightarrow \quad y(x) = I_m(kx) \quad \text{or} \quad y(x) = K_m(kx)$$

$$I_m(x) = \frac{1}{m!} \left(\frac{x}{2} \right)^m + \dots \quad , \quad x \ll 1$$

$$K_0(x) = -\ln(x) + \dots \quad , \quad K_m(x) = \frac{1}{2}(m-1)! \left(\frac{x}{2} \right)^{-m} + \dots \quad , \quad m \geq 1$$

$$I_m(x) \rightarrow \frac{e^x}{\sqrt{2\pi x}} \quad , \quad K_m(x) \rightarrow e^{-x} \sqrt{\frac{\pi}{2x}} \quad , \quad x \gg m + 1$$

$$I_m(x) = i^{-m} J_m(ix) \quad , \quad K_m(x) = \frac{\pi}{2} i^{m+1} \left[J_m(ix) + i Y_m(ix) \right]$$

$$I_{-m}(x) = I_m(x) \quad , \quad K_{-m}(x) = K_m(x)$$