

The Wave Equation Green's Function (DWL 4/22/20)

The electrostatic potential $\Phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ satisfy the inhomogeneous wave equations

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\frac{\rho(\mathbf{x}, t)}{\epsilon_0} \quad (1)$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{x}, t) \quad , \quad (2)$$

given as eqs. (6.14) and (6.15) in Jackson. Their solutions can be written in terms of the Green's function, $G(\mathbf{x}, t)$, satisfying an inhomogeneous version of the wave equation with a 4-dimensional Dirac delta function as its source

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{x}, t) = \delta(t) \delta(\mathbf{x}) \quad . \quad (3)$$

$G(\mathbf{x}, t)$ is the response of the system to a point-impulse delivered at $t = 0$. Here we use the traditional definition of a generic Green's function differing from Jackson's version, eq. (6.41), by a factor -4π . This means that the Green's function we find will need to be multiplied by -4π to obtain $G^{(+)}$ given by eq. (6.44).

We approach the problem using a 4-d Fourier transform¹

$$G(\mathbf{x}, t) = \frac{1}{(2\pi)^2} \int \tilde{G}(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} d^3k d\omega \quad . \quad (4)$$

Fourier transforming eq. (3) yields the algebraic equation

$$\left(\frac{\omega^2}{c^2} - k^2 \right) \tilde{G}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2} \quad , \quad (5)$$

where the right hand side comes from the 4-d transform of the 4-d Dirac delta function. This algebraic equation can be immediately solved to yield the Green's function in Fourier space

$$\tilde{G}(\mathbf{k}, \omega) = \frac{c^2}{(2\pi)^2} \frac{1}{\omega^2 - c^2 k^2} \quad , \quad (6)$$

whose inverse Fourier transform yields the Green's function we seek

$$G(\mathbf{x}, t) = \frac{c^2}{(2\pi)^4} \int \frac{e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}}{\omega^2 - c^2 k^2} d^3k d\omega \quad . \quad (7)$$

We perform this 4-d integral by doing the frequency integral first to obtain

$$F(\mathbf{k}, t) = \int_{\mathcal{L}} \frac{e^{-i\omega t}}{(\omega - ck)(\omega + ck)} d\omega \quad , \quad (8)$$

¹We use the symmetric normalization whereby every transform includes a pre-factor $(2\pi)^{-1/2}$ in both the forward and inverse directions. Combing four transforms raises this pre-factor to the fourth power, giving us $(2\pi)^{-2}$ out front. The inverse includes another factor $(2\pi)^{-2}$ to resulting in the overall $(2\pi)^{-4}$ in eq. (7). Naturally this expression would be the same regardless of our choice of normalization.

where \mathcal{L} is the path used in the inverse Fourier transform. Since the integrand contains poles at $\omega = \pm ck$ we cannot do the integral along the real ω axis. Instead \mathcal{L} must go from $\omega = -\infty$ to $\omega = +\infty$ along some path either above or beneath the poles, as shown in fig. 1. Choosing different paths will result in different functions $F(\mathbf{k}, t)$ and thus different Green's functions. Every version will satisfy the wave equation, eq. (3), but with different initial or final conditions.

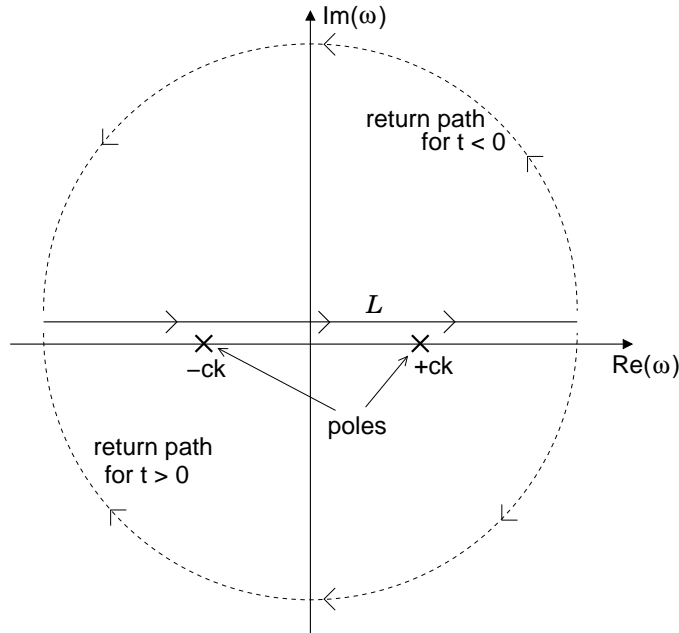


Figure 1: The integration path used to perform the inverse Fourier transform is shown in complex frequency space. The horizontal path \mathcal{L} is shown as a solid line, with arrows indicating direction. Poles at $\omega = \pm ck$ are indicated by crosses on the real axis. Two return paths are shown as dashed arcs. They are labelled according to the condition under which they make no contribution to the integral, provided they are expanded to $|\omega| \rightarrow \infty$ along with the ends of the horizontal path.

The path \mathcal{L} is not closed but we *might* be able to add a “return path” either far above or far below the real axis, as indicated by dashed arcs in fig. 1. If this additional piece adds exactly zero then the closed contour integral will equal the inverse Fourier transform along \mathcal{L} alone. The advantage of using the closed contour is that the integral can be easily done using residues. The exponential in the integrand is

$$e^{-i\omega t} = e^{\omega_i t} e^{-i\omega_r t} \quad , \quad (9)$$

where $\omega_i = \text{Im}(\omega)$ and $\omega_r = \text{Re}(\omega)$. Provided $\omega_i t < 0$ the leading exponential will vanish when the return path is taken far away from the real axis (i.e. $|\omega| \rightarrow \infty$). Only under this condition can we evaluate the inverse Fourier transform using residues. It can be computed by summing the residues of all poles inside the newly closed contour. If the contour encloses no poles then $F(\mathbf{k}, t) = 0$.

Following the above reasoning, when $t < 0$ the path may be closed above ($\omega_i > 0$) thereby enclosing any poles lying above the path \mathcal{L} . We are trying here to compute a Green's function providing the *response* to an impulse delivered at $t = 0$. This means we want $G(\mathbf{x}, t) = 0$ for $t < 0$ so that response comes only *following* the impulse. This gives rise to the term *retarded Green's*

function or causal Green's function. To obtain it we choose the integration path \mathcal{L} above the real axis, and thus above both poles² as shown in fig. 1. In that case the upward closure, for $t < 0$, will enclose no poles giving $F(\mathbf{k}, t) = 0$ and thus $G(\mathbf{x}, t) = 0$ as we wish. For $t > 0$, on the other hand, the path can be closed below the real axis ($\omega_i < 0$) enclosing poles at both $\omega = +ck$ and $\omega = -ck$. We then have

$$F(\mathbf{k}, t) = -2\pi i \sum \text{Res} = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right] = -2\pi \frac{\sin(ckt)}{ck} . \quad (10)$$

We may use this integral in eq. (7) to obtain an expression for the retarded Green's function

$$G(\mathbf{x}, t) = -\Theta(t) \frac{c}{(2\pi)^3} \int \sin(ckt) e^{i\mathbf{k}\cdot\mathbf{x}} \frac{d^3k}{k} , \quad (11)$$

where the Heaviside function $\Theta(t)$ takes care to the fact that $F(\mathbf{k}, t) = 0$ for $t < 0$. It still remains to perform the 3-d integral over \mathbf{k} space to obtain the spatial dependence of the Green's function. Since the equation itself, eq. (3), is isotropic, we expect the result to be independent of angle, and to depend only on distance, $|\mathbf{x}| = r$. This is not, however, clear from the expression as it stands. To make it clearer we substitute the vector \mathbf{k} with a re-scaled version

$$\mathbf{k} = \frac{1}{r} \mathbf{q} , \quad d^3k = \frac{1}{r^3} d^3q . \quad (12)$$

Making this substitution in eq. (11) gives

$$G(\mathbf{x}, t) = -\Theta(t) \frac{c}{(2\pi)^3 r^2} \underbrace{\int \sin\left(\frac{ct}{r} q\right) e^{i\mathbf{q}\cdot\hat{\mathbf{r}}} \frac{d^3q}{q}}_{s(ct/r)} , \quad (13)$$

where $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$ is the radial unit vector. The integral appear to depend on direction through $\hat{\mathbf{r}}$, however, this is not the case. Any change in the direction of $\hat{\mathbf{r}}$ can be accommodated by a corresponding rotation of the \mathbf{q} coordinates leaving the integral unchanged. We are thus left with a single scalar function of a scalar variable

$$s(x) = \int \frac{\sin(qx) e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}}{q} d^3q , \quad (14)$$

defining our Green's function.

The triple integral in eq. (14) is not trivial, but I do it below just to show that I'm being honest. For those who trust me, or at least my integration abilities, I quote the result:

$$s(x) = 2\pi^2 \left[\delta(x-1) - \delta(x+1) \right] . \quad (15)$$

The Heaviside function will eliminate the second Dirac delta function, whose support is at $t = -r/c < 0$, and will be unity when multiplied by the first, whose support is at $t = r/c > 0$

$$\begin{aligned} \Theta(t) s(ct/r) &= 2\pi^2 \Theta(t) \left[\delta(ct/r - 1) - \delta(ct/r + 1) \right] \\ &= 2\pi^2 \delta(ct/r - 1) = \frac{2\pi^2 r}{c} \delta(t - r/c) \end{aligned} \quad (16)$$

²Had we instead chosen to have \mathcal{L} pass beneath all the poles then $G(\mathbf{x}, t)$ would vanish for all $t > 0$. This is an "advanced" Green's function, where the "response" occurs *before* the cause. This is not terribly physical, but it is a mathematical possibility. The remaining possibility would be to have \mathcal{L} pass above one and beneath the other.

after making use of property 5 of the Dirac delta function from Jackson's §1.2. Using this in eq. (13) gives the Green's function

$$G(\mathbf{x}, t) = -\frac{1}{4\pi} \delta(t - r/c) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \delta\left(t - |\mathbf{x}|/c\right) \quad , \quad (17)$$

after replacing r with $|\mathbf{x}|$ to make clear the dependence on position. This is the Green's function for the scalar wave equation in three dimensions. Convolution with it yields the inverse of the operator

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad .$$

The wave equation Green's function in eq. (17) somewhat resembles the more familiar Green's function for the Laplacian alone

$$\mathcal{G}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \quad . \quad (18)$$

Integrating the governing equation, eq. (3), from $t = -\infty$ to $+\infty$ gives the equation governing \mathcal{G} : the solution from a persistent point source. Integrating the Green's function from (17) gives

$$\int_{-\infty}^{\infty} G(\mathbf{x}, t) dt = -\frac{1}{4\pi|\mathbf{x}|} \overbrace{\int_{-\infty}^{\infty} \delta\left(t - |\mathbf{x}|/c\right) dt}^{=1} = \mathcal{G}(\mathbf{x}) \quad . \quad (19)$$

This is how the two Green's functions are related. The factor $-1/4\pi$ is present in both due to the fact that $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$. This is the factor Jackson chooses to exclude defining a so-called Green's function that produced -4π times a 4-d delta function, rather than the delta function alone, as tradition dictates. Multiplying eq. (17) by -4π returns $G^{(+)}$ given by eq. (6.44) in Jackson, as we predicted it would.

It is worth taking a good look at the result given in eq. (17). It vanishes except on the expanding spherical shell $|\mathbf{x}| = ct$ where it is a surface delta-function. Its amplitude decreases inversely with distance as the sphere expands over time. It originates from the origin at $t = 0$, which is the impulse on the right hand side of eq. (3). While this all seems very natural, it is notable that the versions in 1 or 2 dimensions do not share these properties.

Having found the Green's function it is a very simple matter to use it to write the complete general solutions to eqs. (1) and eqs. (2). Using it in the first gives the electrostatic potential

$$\begin{aligned} \Phi(\mathbf{x}, t) &= -\int G(\mathbf{x} - \mathbf{x}', t - t') \frac{\rho(\mathbf{x}', t')}{\epsilon_0} d^3x' dt' \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta\left[(t - t') - |\mathbf{x} - \mathbf{x}'|/c\right] d^3x' dt' \quad , \end{aligned} \quad (20)$$

where the negative sign in front of the first integral comes from the negative sign on the source term of eq. (1). The Dirac delta function permits us to perform the dt' integral with ease. To do so it helps to manipulate its argument

$$\delta\left[(t - t') - |\mathbf{x} - \mathbf{x}'|/c\right] = \delta\left[(t - |\mathbf{x} - \mathbf{x}'|/c) - t'\right] \quad ,$$

to make clear what will replace t' upon integration. We replace it in the charge density with $t - |\mathbf{x} - \mathbf{x}'|/c$ known as the *retarded time*. This the time in the past from which an effect occurring at position \mathbf{x}' would reach the field point \mathbf{x} at time t . Performing the time integral yields the electrostatic potential

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3x' . \quad (21)$$

This is the same as the upper portion of eq. (6.48) of Jackson, although with less compressed notation. The same Green's function can be used for eq. (2) to obtain the vector potential

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3x' . \quad (22)$$

Equation (21) is clearly related to Coulomb's law. If the charge density were genuinely time-independent, $\rho(\mathbf{x})$, then eq. (21) becomes identical with the classic time-independent version of Coulomb's law

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' , \quad (23)$$

given by eq. (1.17) in Jackson. This states that the potential at a point \mathbf{x} includes contributions from all charges weighted inversely by distance. The time-dependent version in eq. (21) says the same thing except that contribution to a point \mathbf{x} at a time t comes from the charges at \mathbf{x}' at the retarded times, rather than the present time.

If the charge density varies in time, but sufficiently slowly, then we can use a “time-dependent static” version of Coulomb's law. By “sufficiently slowly” we mean slowly enough that ρ at the present time, t , is almost the same as it was at the retarded time. In other words, the time taken for light to make the trip from \mathbf{x}' to \mathbf{x} is so brief that ρ has barely changed. A simple way to make this approximation is to formally take the limit $c \rightarrow \infty$ in eq. (21). In so doing the retarded time becomes the present time, $t - |\mathbf{x} - \mathbf{x}'|/c \rightarrow t$, and we obtain a version of Coulomb's law where all charges affect a given point *instantaneously*:

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' . \quad (24)$$

The potential at any time depends on the charges only at that same time. Aside from this, expression (24) is identical with (23). By the same argument the vector potential under this assumption is

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' , \quad (25)$$

matching eq. (5.32) from Jackson.

All the results obtained in Chapters 1–5 thus apply for time-varying charges and current provided only that they vary slowly enough that light-travel times are small enough to ignore. This is the “static” limit. The more complete set of Maxwell's equations, including the displacement current term in Ampère's law, is required only when this assumption is violated — only when charges and/or currents vary on time scales short or comparable to light-travel times across the system of interest.

Doing the triple integral

To perform the triple integral in eq. (14) we define the axes of the \mathbf{q} to align one coordinate, q_1 , along the $\hat{\mathbf{r}}$ direction. We denote by q_\perp the distance perpendicular to that axis so the integral becomes

$$s(x) = \int \frac{\sin(qx) e^{iq_1}}{q} d^3q = 2\pi \int \frac{\sin(x\sqrt{q_1^2 + q_\perp^2}) e^{iq_1}}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp dq_1 . \quad (26)$$

We first perform first the dq_\perp integral

$$\begin{aligned} A(x) &= \int_0^\infty \frac{\sin(x\sqrt{q_1^2 + q_\perp^2})}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp \\ &= -\frac{i}{2} \int_0^\infty \frac{\exp[ix\sqrt{q_1^2 + q_\perp^2}]}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp + \frac{i}{2} \int_0^\infty \frac{\exp[-ix\sqrt{q_1^2 + q_\perp^2}]}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp . \end{aligned} \quad (27)$$

None of the integrals converge if x is real. We will perform each of the final two versions with complex x , and then analytically continue the results onto the axis of real x . We take $\text{Im}(x) > 0$ for the first and $\text{Im}(x) < 0$ for the second, so the exponential decays in each case as $q_\perp \rightarrow \infty$. The first integral is

$$\int_0^\infty \frac{\exp[ix\sqrt{q_1^2 + q_\perp^2}]}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp = \frac{1}{ix} \exp[ix\sqrt{q_1^2 + q_\perp^2}] \Big|_{q_\perp=0}^\infty = \frac{i}{x} e^{ix|q_1|} , \quad (28)$$

where the upper limit vanishes since $\text{Im}(x) > 0$. The second is

$$\int_0^\infty \frac{\exp[-ix\sqrt{q_1^2 + q_\perp^2}]}{\sqrt{q_1^2 + q_\perp^2}} q_\perp dq_\perp = \frac{1}{-ix} \exp[-ix\sqrt{q_1^2 + q_\perp^2}] \Big|_{q_\perp=0}^\infty = -\frac{i}{x} e^{-ix|q_1|} , \quad (29)$$

where the upper limit vanishes since $\text{Im}(x) < 0$. Using these in eq. (27) gives

$$A(x) = \frac{1}{2x} e^{ix|q_1|} + \frac{1}{2x} e^{-ix|q_1|} = \frac{\cos(x|q_1|)}{x} = \frac{\cos(xq_1)}{x} , \quad (30)$$

where the terms are analytically continued onto real x axis to be combined into a cosine. Since q_1 is taken to be real, and cosine is even in its argument, the absolute value can be dropped in the final expression. Putting this into the remaining integral, eq. (26) gives

$$\begin{aligned} s(x) &= \frac{2\pi}{x} \int_{-\infty}^\infty e^{iq_1} \cos(xq_1) dq_1 = \frac{\pi}{x} \int_{-\infty}^\infty \overbrace{e^{i(1+x)q_1}}^{2\pi\delta(1+x)} dq_1 + \frac{\pi}{x} \int_{-\infty}^\infty \overbrace{e^{i(1-x)q_1}}^{2\pi\delta(1-x)} dq_1 \\ &= \frac{2\pi^2}{x} \delta(1+x) + \frac{2\pi^2}{x} \delta(1-x) = -2\pi^2 \delta(x+1) + 2\pi^2 \delta(x-1) , \end{aligned} \quad (31)$$

where the remaining integrals are identified as one-dimensional Dirac-delta functions. This is the result already quoted in eq. (15).