## Notes on radiation (DWL 5/16/16)

The electrostatic potential $\Phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ satisfy the inhomogeneous wave equations

$$
\begin{align*}
& \left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \Phi=-\frac{\rho(\mathbf{x}, t)}{\epsilon_{0}}  \tag{1}\\
& \left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \mathbf{A}=-\mu_{0} \mathbf{J}(\mathbf{x}, t) \tag{2}
\end{align*}
$$

given as eqs. (6.14) and (6.15) in Jackson. Their solutions can be written in terms of the retarded Green's function, $G^{(+)}$in eq. (6.44), leading to

$$
\begin{align*}
\Phi(\mathbf{x}, t) & =\frac{1}{4 \pi \epsilon_{0}} \int \frac{1}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} \rho\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime}  \tag{3}\\
\mathbf{A}(\mathbf{x}, t) & =\frac{\mu_{0}}{4 \pi} \int \frac{1}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} \mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime} \tag{4}
\end{align*}
$$

These appear as eq. (6.48), but here we have made the dependences on $\mathbf{x}^{\prime}$ and $t$ explicit.
We now assume that the charge density $\rho\left(\mathbf{x}^{\prime}, t\right)$ and current density $\mathbf{J}\left(\mathbf{x}^{\prime}, t\right)$, both vanish outside a sphere $\left|\mathbf{x}^{\prime}\right|=a$, for some radius $a$. We consider a field point outside this sphere, $|\mathbf{x}|>a$, and perform multipole expansions of eqs. (3) and (4). The first step in this expansion is

$$
\begin{equation*}
\frac{1}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}=\frac{1}{r}\left(1-2 \frac{\mathbf{x} \cdot \mathbf{x}^{\prime}}{r^{2}}+\frac{\left|\mathbf{x}^{\prime}\right|^{2}}{r^{2}}\right)^{-1 / 2}=\frac{1}{r}+\frac{\mathbf{x} \cdot \mathbf{x}^{\prime}}{r^{3}}+\cdots \tag{5}
\end{equation*}
$$

Inserting this first into eq. (3) yields

$$
\begin{align*}
\Phi(\mathbf{x}, t)= & \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r} \int \rho\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime} \\
& \quad+\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}}}{r^{2}} \cdot \int \mathbf{x}^{\prime} \rho\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime}+\cdots \tag{6}
\end{align*}
$$

Had the charge density been time-independent, this would match the monopole and dipole terms in our traditional multipole expansion. If it does depend on time, however, the spatial coordinate, $\mathbf{x}^{\prime}$ appears in two places, and the integral becomes somewhat complicated. To simplify this we expand the temporal argument in a fashion similar to the expansion in eq. (5)

$$
\begin{equation*}
t-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c}=t-\frac{r}{c}\left(1-2 \frac{\mathbf{x} \cdot \mathbf{x}^{\prime}}{r^{2}}+\frac{\left|\mathbf{x}^{\prime}\right|^{2}}{r^{2}}\right)^{+1 / 2}=\underbrace{t-\frac{r}{c}}_{t_{r}}+\hat{\mathbf{r}} \cdot \frac{\mathbf{x}^{\prime}}{c}+\cdots \tag{7}
\end{equation*}
$$

where we identify the first two terms as the retarded time

$$
\begin{equation*}
t_{r}=t-\frac{r}{c} \tag{8}
\end{equation*}
$$

This is the time a signal must leave the origin to reach the field point $\mathbf{x}$ at time $t$. The next term in eq. (7), $r^{\prime} / c$, the time taken for the signal to reach a point $\mathbf{x}^{\prime}$, inside the charge/current distribution, from the origin. We will assume $a \ll r$, so this time is far smaller than $t_{r}$. Introducing this into the temporal argument and expanding in this small ratio yields

$$
\begin{equation*}
\rho\left(\mathbf{x}^{\prime}, t-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c}\right)=\rho\left(\mathbf{x}^{\prime}, t_{r}\right)+\left.\hat{\mathbf{r}} \cdot \frac{\mathbf{x}^{\prime}}{c} \frac{\partial \rho}{\partial t}\right|_{t=t_{r}}+\cdots . \tag{9}
\end{equation*}
$$

Unlike the original version of retarded time, this quantity $t_{r}$ defined by eq. (8), is independent of $\mathbf{x}^{\prime}$, making spatial integration far simpler. Introducing the expansion, eq. (9), into each of the integrals in eq. (6), and performing those integrals yields

$$
\begin{equation*}
\Phi(\mathbf{x}, t)=\underbrace{\frac{1}{4 \pi \epsilon_{0}} \frac{q\left(t_{r}\right)}{r}}_{\text {i. }}+\underbrace{\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}}}{r c} \cdot \dot{\mathbf{p}}\left(t_{r}\right)}_{\text {ii. }}+\underbrace{\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}}}{r^{2}} \cdot \mathbf{p}\left(t_{r}\right)}_{\text {iii. }}+\cdots, \tag{10}
\end{equation*}
$$

where we have introduced the moments

$$
\begin{align*}
q\left(t_{r}\right) & =\int \rho\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}  \tag{11}\\
\mathbf{p}\left(t_{r}\right) & =\int \mathbf{x}^{\prime} \rho\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime} \tag{12}
\end{align*}
$$

each evaluated at the retarded time $t_{r}$. The three terms, labelled i. ii. and iii. constitute all terms up to order $r^{\prime} / c t_{r} \sim a / r$. Beyond these are terms involving the quadrupole, $\sim a^{2} / r^{2}$, and higher order in $a / r$. Term ii. arose from the manipulations arising from the second term in expansion (7)

$$
\begin{equation*}
\int\left[\left.\hat{\mathbf{r}} \cdot \frac{\mathbf{x}^{\prime}}{c} \frac{\partial \rho}{\partial t}\right|_{t=t_{r}}\right] d^{3} x^{\prime}=\frac{\hat{\mathbf{r}}}{c} \cdot\left(\frac{\partial}{\partial t} \int \mathbf{x}^{\prime} \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}\right)_{t=t_{r}}=\frac{\hat{\mathbf{r}}}{c} \cdot \dot{\mathbf{p}}\left(t_{r}\right) \tag{13}
\end{equation*}
$$

Equation (10) includes the traditional monopole (i.) and dipole (iii.) terms, albeit evaluated at an earlier time. Term ii. has a radial dependence, $r^{-1}$, resembling a monopole, but depends on the dipole moment of the charge distribution - actually its time-derivative. This term could occur even when the charge distribution was entirely charge neutral, $q=0$. Such a charge distribution must include equal parts positive and negative charge density, which cancel out when integrated. The first term in eq. (6) performs the spatial integration so that points further from the field point, $\mathbf{x}$, are evaluated at an earlier time. If that region was dominated by positive charge, and the amplitudes of both signs happened to be increasing in time $(\partial|\rho| / \partial t>0)$ then the earlier positive (far) contribution will fail to cancel the nearer negative contribution, which is evaluated slightly later. As a result there is an overall negative contribution to the first term in eq. (6). This mimics a negative monopole, but it arose because $\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}<0$.

When performing the more conventional multi-pole expansion we tacitly assumed a static charge distribution and therefore omitted term, ii. When it is present, term ii. falls off with a lower power of distance than does term iii. ( $r^{-1}$ compared to $r^{-2}$ ) so it would seem term ii. will always dominate just as the monopole term typically dominates the dipole term. This is not strictly true in this case. To make the comparison more carefully we define a time-scale $\tau$ which characterizes the time over which the charge distribution changes. In particular we assume that, at least in typical cases,

$$
|\dot{\mathbf{p}}| \sim \frac{|\mathbf{p}|}{\tau}
$$

Using this definition in a ratio of those two terms yields

$$
\begin{equation*}
\frac{\text { term ii. }}{\text { term iii. }} \sim \frac{|\dot{\mathbf{p}}| / r c}{|\mathbf{p}| / r^{2}} \sim \frac{r}{c \tau} \tag{14}
\end{equation*}
$$

This means that while term ii. does fall off more slowly with increasing radius, it only dominates term iii. outside a radius $c \tau$. This is the distance traveled by light during the time taken for the charge distribution to change.

This distance, $c \tau$, forms a second characteristic length we must consider in doing our expansion. ${ }^{1}$ The region well inside this

$$
a \ll r \ll c \tau
$$

is known as the static zone. In this region term iii. dominates term ii. and the latter can be neglected. Doing so returns our traditional multipole expansion. It arises, however, not from assuming the charge distribution to be static, but simply assuming it changes "slowly". In this case slowly means the time scale of evolution $\tau \ll r / c$, the time taken for light to reach the field point $\mathbf{x}$. In fact, that is what it truly means to be "static" in electrostatics: evolution must occur slowly compared to light-travel times.

Far outside the static zone

$$
\begin{equation*}
r \gg a, c \tau \tag{15}
\end{equation*}
$$

lies the so-called radiation zone. In this region we can ignore term iii. with respect to term ii. The result will be different from the multipole expansion we have previously considered. Since we have already treated the static zone extensively (without exactly knowing we were doing so) we hence forth assume we are the radiation zone, and use eq. (15) the eliminate small terms.

## Life in the radiation zone

Before we begin we consider the possible time-dependence of the net charge $q$. The defining integral, (11), must be done over the ball $\mathcal{B}_{a}$, for which $\left|\mathbf{x}^{\prime}\right|<a$. The time derivative of this integral is

$$
\begin{equation*}
\dot{q}=\int_{\mathcal{B}_{a}} \frac{\partial \rho}{\partial t} d^{3} x^{\prime}=-\int_{\mathcal{B}_{a}} \nabla^{\prime} \cdot \mathbf{J} d^{3} x^{\prime}=-\oint_{\partial \mathcal{B}_{a}} \mathbf{J} \cdot d \mathbf{a}^{\prime}=0, \tag{16}
\end{equation*}
$$

after using first, charge continuity $\partial \rho / \partial t=-\nabla \cdot \mathbf{J}$, and then the fact that charge density and current density both vanish on there surface $\partial \mathcal{B}_{a}$, i.e. the sphere $\left|\mathbf{x}^{\prime}\right|=a$. Thus, a consequence of charge conservation is that the monopole moment, $q$, does not vary in time. It is possible for all other multipole moments to vary - just not the monopole moment. We henceforth omit its time dependence.

In the radiation zone we may drop term iii. from eq. (10), to obtain

$$
\begin{equation*}
\Phi(\mathbf{x}, t)=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{r}+\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}}}{r c} \cdot \dot{\mathbf{p}}\left(t_{r}\right) . \tag{17}
\end{equation*}
$$

[^0]To find the electric field we must take a gradient of this expression. In so doing we use the vector identity

$$
\begin{align*}
\nabla\left[\frac{\hat{\mathbf{r}}}{r c} \cdot \dot{\mathbf{p}}\left(t_{r}\right)\right] & =\dot{\mathbf{p}} \cdot \nabla\left(\frac{\hat{\mathbf{r}}}{r c}\right)+\frac{\hat{\mathbf{r}}}{r c} \cdot \nabla \dot{\mathbf{p}}+\frac{\hat{\mathbf{r}}}{r c} \times(\nabla \times \dot{\mathbf{p}})+\dot{\mathbf{p}} \times \underbrace{\left[\nabla \times\left(\frac{\hat{\mathbf{r}}}{r c}\right)\right]}_{=0} \\
& =\dot{\mathbf{p}} \cdot \nabla\left(\frac{\hat{\mathbf{r}}}{r c}\right)+\frac{1}{r c} \frac{\partial \dot{\mathbf{p}}}{\partial r}+\frac{\hat{\mathbf{r}}}{r c} \times(\nabla \times \dot{\mathbf{p}}) \tag{18}
\end{align*}
$$

The fourth term in the intermediate expression is easily seen to vanish since $\hat{\mathbf{r}} / r=\nabla \ln (r)$, has zero curl. The second and third might seem to vanish since they involve spatial derivatives of a multipole moment. This moment is, however, evaluated at an argument, $t_{r}=t-r / c$, which does depend on position. The second term therefore becomes

$$
\begin{equation*}
\frac{1}{r c} \frac{\partial \dot{\mathbf{p}}}{\partial r}=\frac{1}{r c} \ddot{\mathbf{p}} \frac{\partial t_{r}}{\partial r}=-\frac{1}{r c^{2}} \ddot{\mathbf{p}}\left(t_{r}\right) . \tag{19}
\end{equation*}
$$

The curl in the third term is

$$
\begin{equation*}
\nabla \times \dot{\mathbf{p}}=\nabla t_{r} \times \ddot{\mathbf{p}}=-\frac{\hat{\mathbf{r}}}{c} \times \ddot{\mathbf{p}}\left(t_{r}\right) \tag{20}
\end{equation*}
$$

This gives, in combination,

$$
\begin{equation*}
\nabla\left[\frac{\hat{\mathbf{r}}}{r c} \cdot \dot{\mathbf{p}}\left(t_{r}\right)\right]=\dot{\mathbf{p}}\left(t_{r}\right) \cdot \nabla\left(\frac{\hat{\mathbf{r}}}{r c}\right)-\frac{\hat{\mathbf{r}}}{r c^{2}}\left[\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}\left(t_{r}\right)\right] . \tag{21}
\end{equation*}
$$

Since $\nabla(\hat{\mathbf{r}} / r c) \sim 1 / r^{2} c$, the first term in this expression can be shown to be negligible compared to the second in the radiation zone - it is smaller by $c \tau / r \ll 1$.

With all of these calculations in place we can evaluate the gradient of eq. (17)

$$
\begin{equation*}
-\nabla \Phi=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{r^{2}} \hat{\mathbf{r}}+\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}}{r c^{2}} \hat{\mathbf{r}}+\cdots \tag{22}
\end{equation*}
$$

The first term is obviously the monpole contribution to the electric field, falling off as $r^{-2}-$ Coulomb's inverse-square law. The second term is a radiative contribution which falls off only as $r^{-1}$, more slowly even than the monopole term. In order to find the full electric field, however, we must evaluate the vector potential.

## Dipole radiation

Introducing expansion (5) into eq. (4) yields

$$
\begin{align*}
\mathbf{A}(\mathbf{x}, t)= & \frac{\mu_{0}}{4 \pi} \frac{1}{r} \int \mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime} \\
& +\frac{\mu_{0}}{4 \pi} \frac{\hat{\mathbf{r}}}{r^{2}} \cdot \int \mathbf{x}^{\prime} \mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{x}-\mathbf{x}^{\prime}\right| / c\right) d^{3} x^{\prime}+\cdots \tag{23}
\end{align*}
$$

We next use expansion (7) to expand each of the current densities about a time $t_{r}$. The result is

$$
\begin{align*}
\mathbf{A}(\mathbf{x}, t)= & \overbrace{\frac{\mu_{0}}{4 \pi} \frac{1}{r} \int \mathbf{J}\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}}^{\mathrm{i} .}+\overbrace{\frac{\mu_{0}}{4 \pi} \frac{\partial}{\partial t}\left[\frac{1}{r c} \int\left(\hat{\mathbf{r}} \cdot \mathbf{x}^{\prime}\right) \mathbf{J}\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}\right]}^{\text {ii. }} \\
& +\underbrace{\frac{\mu_{0}}{4 \pi} \frac{1}{r^{2}} \int\left(\hat{\mathbf{r}} \cdot \mathbf{x}^{\prime}\right) \mathbf{J}\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}}_{\text {iii. }}+\cdots \tag{24}
\end{align*}
$$

where terms beyond the magnetic dipole have been dropped, and the remaining terms have been numbered for future reference. Term i., in which $\mathbf{A} \sim r^{-1}$ appears poised to give rise to a magnetic monopole term, $\mathbf{B} \sim r^{-2}$, so it must have been absent in the static treatment. We rewrite it in index notation using a Kronecker-delta and Einstein's summation convention

$$
\begin{align*}
\int J_{j}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime} & =\int \overbrace{\frac{\partial x_{j}^{\prime}}{\partial x_{k}^{\prime}}}^{\delta_{j k}} J_{k}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}=-\int x_{j}^{\prime} \frac{\partial J_{k}}{\partial x_{k}^{\prime}} d^{3} x^{\prime} \\
& =\int x_{j}^{\prime} \frac{\partial \rho}{\partial t} d^{3} x^{\prime}=\frac{\partial}{\partial t} \int x_{j}^{\prime} \rho\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}=\dot{p}_{j}\left(t_{r}\right) \tag{25}
\end{align*}
$$

after discarding the surface term in the integration-by-parts, and using charge continuity, $\partial \rho / \partial t=$ $-\nabla \cdot \mathbf{J}=-\partial J_{k} / \partial x_{k}^{\prime}$. In truly static cases the electric dipole moment is time-independent and this term will vanish. In the present case, however, it does not.

The next two terms are handled using a similar trick

$$
\begin{align*}
\int x_{i}^{\prime} J_{j}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime} & =\int \overbrace{\frac{\partial x_{j}^{\prime}}{\partial x_{k}^{\prime}}}^{\delta_{j k}^{\prime}} x_{i}^{\prime} J_{k}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}=-\int x_{j}^{\prime} \frac{\partial}{\partial x_{k}^{\prime}}\left[x_{i} J_{k}\left(\mathbf{x}, t_{r}\right)\right] d^{3} x^{\prime} \\
& =-\int x_{j}^{\prime} \underbrace{\left(\frac{\partial x_{i}^{\prime}}{\partial x_{k}^{\prime}}\right)}_{\delta_{i k}} J_{k}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}-\int x_{j}^{\prime} x_{i}^{\prime}\left(\frac{\partial J_{k}}{\partial x_{k}^{\prime}}\right) d^{3} x^{\prime} \\
& =-\int x_{j}^{\prime} J_{i}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}-\frac{\partial}{\partial t} \int x_{j}^{\prime} x_{i}^{\prime} \rho\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime} \tag{26}
\end{align*}
$$

Adding half the right hand side to half the left side yields

$$
\begin{equation*}
\int x_{i}^{\prime} J_{j}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}=\frac{1}{2} \int\left[x_{i}^{\prime} J_{j}\left(\mathbf{x}, t_{r}\right)-x_{j}^{\prime} J_{i}\left(\mathbf{x}, t_{r}\right)\right] d^{3} x^{\prime}-\frac{1}{2} \dot{M}_{i j}\left(t_{r}\right) \tag{27}
\end{equation*}
$$

where we have introduced the second moment of the charge distribution

$$
\begin{equation*}
M_{i j}\left(t_{r}\right)=\int x_{j}^{\prime} x_{i}^{\prime} \rho\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime} \tag{28}
\end{equation*}
$$

which is related to the quadrupole moment. Until now we have ignored this contribution, but here we see that it is of comparable order to the magnetic dipole. In the interest of completeness we ignore it for the moment, and consider only the electric and magnetic dipoles.

Terms ii. and iii. involve tensors of the form (27), but contracted with $\hat{\mathbf{r}}$. Performing the contraction, and omitting the electric quadrupole, yields

$$
\begin{equation*}
\int\left(\hat{r}_{i} x_{i}^{\prime}\right) J_{j}\left(\mathbf{x}, t_{r}\right) d^{3} x^{\prime}=\frac{1}{2} \int\left[\left(\hat{r_{i}} x_{i}^{\prime}\right) J_{j}-x_{j}^{\prime}\left(\hat{r}_{i} J_{i}\right)\right] d^{3} x^{\prime}=\frac{1}{2} \int\left[\left(\mathbf{x}^{\prime} \times \mathbf{J}\right) \times \hat{\mathbf{r}}\right]_{j} d^{3} x^{\prime} \tag{29}
\end{equation*}
$$

Introducing the magnetic dipole moment

$$
\begin{equation*}
\mathbf{m}\left(t_{r}\right)=\frac{1}{2} \int \mathbf{x}^{\prime} \times \mathbf{J}\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime} \tag{30}
\end{equation*}
$$

we find the product

$$
\begin{equation*}
\int\left(\hat{\mathbf{r}} \cdot \mathbf{x}^{\prime}\right) \mathbf{J}\left(\mathbf{x}^{\prime}, t_{r}\right) d^{3} x^{\prime}=\mathbf{m}\left(t_{r}\right) \times \hat{\mathbf{r}} \tag{31}
\end{equation*}
$$

Placing these results into (24) gives a vector potential

$$
\begin{equation*}
\mathbf{A}(\mathbf{x}, t)=\overbrace{\frac{\mu_{0}}{4 \pi} \frac{\dot{\mathbf{p}}\left(t_{r}\right)}{r}}^{\mathrm{i} .}+\overbrace{\frac{\mu_{0} \dot{\mathbf{m}}\left(t_{r}\right) \times \hat{\mathbf{r}}}{4 \pi}}^{r c}+\overbrace{\frac{\mu_{0} \mathbf{m}\left(t_{r}\right) \times \hat{\mathbf{r}}}{4 \pi} \frac{\mathrm{r}}{r^{2}}}^{\mathrm{iii}} . \tag{32}
\end{equation*}
$$

Term iii. is the classic magnetic dipole term. Following the same arguments given above, it will dominate term ii. within the static zone. In that case we obtain a magnetic field from a static dipole, $\mathbf{B} \sim|\mathbf{m}| / r^{3}$, even if the magnetic dipole is slowly changing. In the radiation zone, in contrast, term ii. will dominate.

To complete the electric field arising from the electric dipole, however, we retain only term i. In this case the full electric field becomes

$$
\begin{equation*}
\mathbf{E}_{p}(\mathbf{x}, t)=-\nabla \Phi-\frac{\partial \mathbf{A}}{\partial t}=\frac{1}{4 \pi \epsilon_{0}} \frac{\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}\left(t_{r}\right)}{r c^{2}} \hat{\mathbf{r}}-\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathbf{p}}\left(t_{r}\right)}{r} \tag{33}
\end{equation*}
$$

where we have omitted the monopole term for clarity - it is just the static monopole term since $q$ never changes. Substituting $c^{2}=1 / \epsilon_{0} \mu_{0}$ yields a field

$$
\begin{equation*}
\mathbf{E}_{p}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi} \frac{1}{r}[(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}) \hat{\mathbf{r}}-\ddot{\mathbf{p}}]=\frac{\mu_{0}}{4 \pi} \frac{\left[\ddot{\mathbf{p}}\left(t_{r}\right) \times \hat{\mathbf{r}}\right] \times \hat{\mathbf{r}}}{r} \tag{34}
\end{equation*}
$$

after using the vector identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}=(\mathbf{a} \cdot \mathbf{c}) \mathbf{b}-(\mathbf{b} \cdot \mathbf{c}) \mathbf{a}$
Finally, we compute the magnetic field by taking the curl of term i. from eq. (32),

$$
\begin{equation*}
\mathbf{B}_{p}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi} \nabla \times\left[\frac{\dot{\mathbf{p}}\left(t_{r}\right)}{r}\right]=\frac{\mu_{0}}{4 \pi} \frac{1}{r}[\nabla \times \dot{\mathbf{p}}]+\cdots=-\frac{\mu_{0}}{4 \pi} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{p}}\left(t_{r}\right)}{r c}+\cdots \tag{35}
\end{equation*}
$$

after dropping terms $\sim 1 / r^{2}$, negligible in the radiation zone, and making use of expression (20). The electric and magnetic field take the form

$$
\begin{equation*}
\mathbf{B}_{p}=\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathbf{p}} \times \hat{\mathbf{r}}}{r c} \quad, \quad \mathbf{E}_{p}=\frac{\mu_{0}}{4 \pi} \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{r}=c \mathbf{B}_{p} \times \hat{\mathbf{r}} \tag{36}
\end{equation*}
$$

This makes clear that both electric and magnetic field are perpendicular to $\hat{\mathbf{r}}$ and to one-another. Their magnitudes are $\left|\mathbf{B}_{p}\right|=\left|\mathbf{E}_{p}\right| / c$. These are clearly consistent with an electromagnetic wave. This is not, however, a plane-wave. The Poynting vector

$$
\begin{equation*}
\mathbf{S}(\mathbf{x}, t)=\frac{1}{\mu_{0}} \mathbf{E}_{p} \times \mathbf{B}_{p}=\frac{c}{\mu_{0}}\left|\mathbf{B}_{p}\right|^{2} \hat{\mathbf{r}}=\frac{\mu_{0}}{16 \pi^{2} c r^{2}}\left|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}\left(t_{r}\right)\right|^{2} \hat{\mathbf{r}}, \tag{37}
\end{equation*}
$$

is purely radial. The energy flux is always outward ( $\mathbf{r} \cdot \mathbf{S}>0$ ) regardless of how the charges move. This is a direct consequence of choosing the retarded Green's function $G^{(+)}$to produce eqs. (3) and (4).

Denoting by $\theta_{p r}$ the angle between $\ddot{\mathbf{p}}$ and $\hat{\mathbf{r}}$, we may express the Poynting vector, eq. (37) as

$$
\begin{equation*}
\mathbf{S}(\mathbf{x}, t)=\frac{\mu_{0}}{16 \pi^{2} c r^{2}} \ddot{p}^{2}\left(t_{r}\right) \sin ^{2} \theta_{p r} \hat{\mathbf{r}} . \tag{38}
\end{equation*}
$$

Integrating over a sphere gives the total power crossing radius $R$

$$
\begin{equation*}
P(R, t)=\oint_{r=R} \mathbf{S} \cdot d \mathbf{a}=\frac{\mu_{0}}{16 \pi^{2} c} \ddot{p}^{2}(t-R / c) \underbrace{\oint \sin ^{2} \theta_{p r} d \Omega}_{=8 \pi / 3}=\frac{\mu_{0}}{6 \pi c} \ddot{p}^{2} \tag{39}
\end{equation*}
$$

The factor $\ddot{p}$ is evaluated at the earlier time whose light has just reached the surface $r=R$ at time $t$. The power emitted from a particular time propagates to all radii without diminishing.

## Magnetic dipole radiation

The results above concern the most common form of radiation: radiation from a time-varying electric dipole, $\mathbf{p}(t)$. Because we are very close to the result, we consider a case where there is no electric dipole $(\mathbf{p}=0)$, but there is a magnetic dipole. In this case we retain only term ii. in eq. (32), which dominates iii. in the radiation zone, yielding

$$
\begin{equation*}
\mathbf{A}_{m}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi} \frac{\dot{\mathbf{m}}\left(t_{r}\right) \times \hat{\mathbf{r}}}{r c} \tag{40}
\end{equation*}
$$

The electrostatic potential entirely vanishes, leaving the electric field

$$
\begin{equation*}
\mathbf{E}_{m}(\mathbf{x}, t)=-\frac{\partial \mathbf{A}_{m}}{\partial t}=-\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathbf{m}} \times \hat{\mathbf{r}}}{r c} \tag{41}
\end{equation*}
$$

The magnetic field is a bit trickier

$$
\begin{align*}
\mathbf{B}_{m}(\mathbf{x}, t) & =\nabla \times \mathbf{A}_{m}=\frac{\mu_{0}}{4 \pi c} \nabla \times\left[\frac{\dot{\mathbf{m}}\left(t_{r}\right) \times \hat{\mathbf{r}}}{r}\right]=\frac{\mu_{0}}{4 \pi c} \frac{1}{r} \nabla \times\left[\dot{\mathbf{m}}\left(t_{r}\right) \times \hat{\mathbf{r}}\right]+\cdots \\
& =\frac{\mu_{0}}{4 \pi c} \frac{1}{r}[(\hat{\mathbf{r}} \cdot \nabla) \dot{\mathbf{m}}-\hat{\mathbf{r}}(\nabla \cdot \dot{\mathbf{m}})]=\frac{\mu_{0}}{4 \pi c} \frac{1}{r}[\hat{\mathbf{r}} \cdot(\nabla \dot{\mathbf{m}})-\hat{\mathbf{r}} \operatorname{Tr}(\nabla \dot{\mathbf{m}})] \tag{42}
\end{align*}
$$

after dropping all terms $\sim 1 / r^{2}$, negligible in the radiation zone, and introducing the derivative tensor $\nabla \ddot{\mathbf{m}}$. The spatial dependence of $\dot{\mathbf{m}}$ comes from its argument, $t_{r}=t-r / c$. Differentiating w.r.t. this yields a tensor derivative

$$
\begin{equation*}
\nabla \dot{\mathbf{m}}=-\frac{\hat{\mathbf{r}}}{c} \ddot{\mathbf{m}} \tag{43}
\end{equation*}
$$

Using this leads to

$$
\begin{equation*}
\mathbf{B}_{m}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi c^{2}} \frac{1}{r}[-\ddot{\mathbf{m}}+\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}})]=\frac{\mu_{0}}{4 \pi c^{2}} \frac{1}{r}[\hat{\mathbf{r}} \times(\hat{\mathbf{r}} \times \ddot{\mathbf{m}})]=\frac{\hat{\mathbf{r}} \times \mathbf{E}_{m}}{c} . \tag{44}
\end{equation*}
$$

Using this in the Poynting flux gives

$$
\begin{equation*}
\mathbf{S}(\mathbf{x}, t)=\frac{1}{\mu_{0}} \mathbf{E}_{m} \times \mathbf{B}_{m}=\frac{1}{c \mu_{0}}\left|\mathbf{E}_{m}\right|^{2} \hat{\mathbf{r}}=\frac{\mu_{0}}{16 \pi^{2} c^{3} r^{2}}\left|\hat{\mathbf{r}} \times \ddot{\mathbf{m}}\left(t_{r}\right)\right|^{2} \hat{\mathbf{r}} \tag{45}
\end{equation*}
$$

This is very similar to eq. (37), but with $\mathbf{p} \rightarrow \mathbf{m} / c$. For a collection of charges, moving at typical speed $v$, the electric and magnetic dipole moments will compare as $|\mathbf{m}| \sim|\mathbf{p}| v$, where $v$ is the typical velocity of the charges. In this case the electric dipole radiation will exceed the magnetic dipole version by a factor $v / c$.

Finally, we note that the polarization is perpendicular to $\ddot{\mathbf{m}}$. This is not surprising, since the magnetic field aligns with $\ddot{\mathbf{m}}$, and the electric field is perpendicular to that. The radiation power is largest along the plane perpendicular to $\ddot{\mathbf{m}}$, and zero in the direction parallel to it.

## Examples

## I. Dipole antenna

Consider a simple antenna consisting of two isolated conductors, each connected to a different lead from a transmission line. The ends of the transmission line carry currents $\pm I_{a}(t)$, which deposit charge $\pm q_{a}(t)$ onto the two conductors. We assume for the moment that the antenna operates at low enough frequency that the charge distributes itself over each conductor in some manner dependent only on geometry. The result is a dipole moment

$$
\begin{equation*}
\mathbf{p}(t)=q_{a}(t) \ell_{\mathrm{eff}} \hat{\mathbf{1}}, \tag{46}
\end{equation*}
$$

where $\ell_{\text {eff }}$ is an effective separation between charges (i.e. between conductors) and $\hat{l}$ is a unit vector specifying direction of the dipole when $q_{a}>0$. Both $\ell_{\text {eff }}$ and $\hat{l}$, are assumed fixed by the geometry of the antenna, so

$$
\begin{equation*}
\dot{p}(t)=\dot{q}_{a}(t) \ell_{\mathrm{eff}} \hat{\mathbf{l}}=I_{a}(t) \ell_{\mathrm{eff}} \hat{\mathbf{l}} . \tag{47}
\end{equation*}
$$

We further assume the current to be monochromatic with angular frequency $\omega$

$$
\begin{equation*}
I_{a}(t)=I_{0} \sin (\omega t) \quad, \quad \ddot{p}(t)=\omega I_{0} \ell_{\mathrm{eff}} \cos (\omega t) \hat{\mathbf{l}} \tag{48}
\end{equation*}
$$

Performing a time average of $\ddot{p}^{2}$ yields

$$
\begin{equation*}
\left\langle\ddot{p}^{2}\right\rangle=\frac{1}{2} \omega^{2} I_{0}^{2} \ell_{\mathrm{eff}}^{2}=2 \pi^{2} c^{2} I_{0}^{2}\left(\frac{\ell_{\mathrm{eff}}}{\lambda}\right)^{2} \tag{49}
\end{equation*}
$$

where $\lambda=2 \pi c / \omega$ is the wavelength of the radiation in free space. Placing this into an average of the power in (39) yields the average power radiated by the antenna

$$
\begin{equation*}
\langle P\rangle=\frac{\pi}{3} \mu_{0} c I_{0}^{2}\left(\frac{\ell_{\mathrm{eff}}}{\lambda}\right)^{2}=Z_{r}\left\langle I_{a}^{2}\right\rangle\left(\frac{\ell_{\mathrm{eff}}}{\lambda}\right)^{2} \tag{50}
\end{equation*}
$$

where we have defined a characteristic impedance

$$
\begin{equation*}
Z_{r}=\frac{2 \pi}{3} \mu_{0} c=\frac{2 \pi}{3} \sqrt{\frac{\mu_{0}}{\epsilon_{0}}}=790 \Omega \tag{51}
\end{equation*}
$$

proportional to the so-called impedance of free space; we also used the fact that $\left\langle I_{a}^{2}\right\rangle=I_{0}^{2} / 2$.
Our assumption of quasi-static charge distribution was roughly equivalent to assuming $\ell_{\mathrm{eff}} \ll \lambda$, so the mean power will be $\langle P\rangle \ll Z_{r} I_{0}^{2}$. As frequency increases, and $\lambda$ decreases, the power will increase. Eventually, the assumption of quasi-static charge will fail and $\ell_{\text {eff }}$ can be considered to
be frequency dependent. It is very unlikely that rapidly varying charge can ever separate by more than $\ell_{\text {eff }} \sim \lambda$, so even at very high frequencies we expect power to be limited to about $Z_{r} I_{0}^{2}$.

Finally, we can use this average power in eq. (38) to find the average energy flux

$$
\begin{equation*}
\langle\mathbf{S}\rangle=\frac{3}{8 \pi r^{2}} \sin ^{2} \theta_{\ell r}\langle P\rangle \hat{\mathbf{r}} \tag{52}
\end{equation*}
$$

where $\theta_{\ell r}$ is the angle between $\hat{\mathbf{l}}$ and $\hat{\mathbf{r}}$. This shows the well-known distribution of intensity, sometimes called the antenna pattern. From eq. (36) we see that the electromagnetic waves will be polarized along a direction

$$
\begin{equation*}
\hat{\boldsymbol{\epsilon}}=\frac{\hat{\mathbf{r}} \times(\hat{\mathbf{l}} \times \hat{\mathbf{r}})}{\sin \theta_{\ell r}}=\frac{1}{\sin \theta_{\ell r}}(I-\hat{\mathbf{r}} \hat{\mathbf{r}}) \cdot \hat{\mathbf{l}} \tag{53}
\end{equation*}
$$

which is the normalized projection of $\hat{\mathbf{1}}$ onto the plane perpendicular to $\hat{\mathbf{r}}$.

## II. Thomson scattering

The simplest example of a radiating system is a single point particle with charge $q$ and mass $m$. As it follows a path $\mathbf{r}(t)$ through space it has a dipole moment $\mathbf{p}(t)=q \mathbf{r}(t)$. The second time derivative, $\ddot{p}=q \ddot{r}$, is proportional to the acceleration of the charged particle. The charge will radiate if it is subject to some force such as an electric field.

Let us consider this free particle subject to the electric field from a monochromatic incident plane wave

$$
\begin{equation*}
\mathbf{E}_{0}(\mathbf{x}, t)=E_{0} \cos (\mathbf{k} \cdot \mathbf{x}-\omega t) \hat{\boldsymbol{\epsilon}}_{0} \tag{54}
\end{equation*}
$$

which we call the incident wave - it is incident on the free particle. The direction of the electric field, $\hat{\epsilon}_{0}$, is the polarization of the incident wave. The particle will accelerate periodically due to this field, causing it to move periodically about some point, which we designate as the origin. We assume the periodic motion takes the particle some distance from the origin $|\Delta \mathbf{r}| \ll c / \omega$ much smaller than the wavelength of the wave. This means the second derivative of the particle's dipole moment will be

$$
\begin{equation*}
\ddot{\mathbf{p}}(t)=q \overbrace{\left(\frac{q \mathbf{E}_{0}(0, t)}{m}\right)}^{\ddot{\mathbf{r}}(t)}=\frac{q^{2}}{m} E_{0} \cos (\omega t) \hat{\boldsymbol{\epsilon}}_{0} \tag{55}
\end{equation*}
$$

provided it feels only the force from the electric field of the incident wave.
The same plane wave will have a magnetic field directed in the $\pm \hat{\boldsymbol{\epsilon}}_{0} \times \hat{\mathbf{k}}$ direction, with magnitude $B_{0}=E_{0} / c$. This produces a second force on the particle whose magnitude is $q v B_{0}=(v / c) q E_{0}$. Provided the particle's motion is non-relativistic, $v \ll c$, the magnetic force will be be far smaller than the electric force, and eq. (55) will be a very good approximation. This non-relativistic condition will be violated when the incident wave is of very great intensity. If this were the case the non-relativistic dynamics used here are no longer valid either.

Using eq. (55) in eq. (39) gives a radiated power

$$
\begin{equation*}
P=\frac{\mu_{0}}{6 \pi c} \frac{q^{4}}{m^{2}} E_{0}^{2} \cos ^{2}(\omega t) \tag{56}
\end{equation*}
$$

Averaging over a period (or some integral number of them) gives a mean power

$$
\begin{equation*}
\langle P\rangle=\frac{\mu_{0}}{6 \pi c} \frac{q^{4}}{m^{2}} \frac{E_{0}^{2}}{2}=\frac{\mu_{0}}{6 \pi c^{2} \epsilon_{0}} \frac{q^{4}}{m^{2}} \bar{S}_{0}=\frac{\mu_{0}^{2}}{6 \pi} \frac{q^{4}}{m^{2}} \bar{S}_{0}, \tag{57}
\end{equation*}
$$

where $\bar{S}_{0}=\epsilon_{0} c E_{0}^{2} / 2$ is the mean energy flux of the incident plane wave.
Equation (57) admits a simple and suggestive interpretation. A portion of the incident energy flux $\bar{S}_{0}$ is intercepted by a cross sectional area

$$
\begin{equation*}
\sigma_{t}=\frac{\mu_{0}^{2}}{6 \pi} \frac{q^{4}}{m^{2}}=\frac{8 \pi}{3}\left(\frac{\mu_{0} q^{2}}{4 \pi m}\right)^{2}=\frac{8 \pi}{3}\left(\frac{q^{2}}{4 \pi \epsilon_{0} m c^{2}}\right)^{2} \tag{58}
\end{equation*}
$$

and reradiated. That is to say the incident wave is scattered and $\sigma_{t}$ is the scattering cross section. The second and third versions of the same expression replace $\mu_{0}=1 / \epsilon_{0} c^{2}$ to obtain a version of the cross section expressed in terms of a characteristic length

$$
\begin{equation*}
r_{c}=\frac{q^{2}}{4 \pi \epsilon_{0} m c^{2}} \tag{59}
\end{equation*}
$$

known as the classical radius of the particle. The classical radius, and thus the scattering cross section, depend on the particle's charge and mass but not on its actual physical size. Indeed, even a point particle, with zero radius, will have a finite classical radius. The classical radius of the electron is $r_{e}=2.8 \times 10^{-15} \mathrm{~m}$ (almost a million times smaller than an atom); the classical radius of a proton is 1836 times smaller still, and thus far smaller than the proton itself!

The foregoing has shown how a single free charge is able to scatter a portion of an incident plane wave and re-radiate it - scatter it - into all other directions. This process is known as Thomson scattering and $\sigma_{t}$, defined in eq. (58) is the Thomson scattering cross section. The cross section is independent of all parameters other than the mass and charge of the free particle. Particles of low mass, such as electrons, are the most effective scatters.

Any radiation observed to be directed along $\hat{\mathbf{r}}$, toward our field point $\mathbf{x}$, will have been scattered by an angle $\theta_{\text {sc }}$,

$$
\cos \theta_{\mathrm{sc}}=\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} .
$$

The mean energy flux in scattered light will be, according to eq. (38)

$$
\begin{equation*}
\left\langle S_{r}\right\rangle=\frac{3}{8 \pi r^{2}} \sin ^{2} \theta_{p r}\langle P\rangle=\frac{r_{c}^{2}}{r^{2}} \sin ^{2} \theta_{p r} \bar{S}_{0} \tag{60}
\end{equation*}
$$

where $\theta_{p r}$ is the angle between $\hat{\mathbf{r}}$ and the induced dipole. According to eq. (55) direction of the induced dipole is the same as the initial polarization direction $\hat{\boldsymbol{\epsilon}}_{0}$, which must be perpendicular to $\hat{\mathbf{k}}$. The possibilities can be decomposed into two cases: perpendicular to the scattering pane ( $\hat{\epsilon}_{0} \| \hat{\mathbf{r}} \times \hat{\mathbf{k}}$ ), or lying within the scattering plane ( $\hat{\epsilon}_{0} \perp \hat{\mathbf{r}} \times \hat{\mathbf{k}}$ ). In the first case $\hat{\boldsymbol{\epsilon}}_{0}$ is perpendicular to $\hat{\mathbf{r}}$ so $\theta_{p r}=\pi / 2$ and

$$
\begin{equation*}
\left\langle S_{r}^{\perp}\right\rangle=\frac{r_{c}^{2}}{r^{2}} \bar{S}_{0}^{\perp} \tag{61}
\end{equation*}
$$

In the second case $\hat{\boldsymbol{\epsilon}}_{0}$ lies in the same plane with $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$, which themselves make an angle $\theta_{\mathrm{sc}}$. Since $\hat{\boldsymbol{\epsilon}}_{0}$ is $\pi / 2$ from $\hat{\mathbf{k}}$, it must lie $\theta_{p r}=\pi / 2+\theta_{\text {sc }}$ from $\hat{\mathbf{r}}$. The energy flux is therefore

$$
\begin{equation*}
\left\langle S_{r}^{\|}\right\rangle=\frac{r_{c}^{2}}{r^{2}} \bar{S}_{0}^{\|} \cos ^{2} \theta_{\mathrm{sc}} \tag{62}
\end{equation*}
$$

Except for the cases of scattering perfectly forward $\left(\theta_{\mathrm{sc}}=0\right)$ or backward $\left(\theta_{\mathrm{sc}}=\pi\right)$ the intensity of light scattered from in-plane polarization will be less than that of perpendicular polarization.

If the incident light is unpolarized, then it consists of equal parts of both polarizations,

$$
\bar{S}_{0}^{\perp}=\bar{S}_{0}^{\|}=\frac{1}{2} \bar{S}_{0} .
$$

The intensity of scattered light will be

$$
\begin{equation*}
\left\langle S_{r}\right\rangle=\left\langle S_{r}^{\perp}\right\rangle+\left\langle S_{r}^{\|}\right\rangle=\frac{r_{c}^{2}}{r^{2}} \bar{S}_{0}\left(1-\frac{1}{2} \sin ^{2} \theta_{\mathrm{sc}}\right) . \tag{63}
\end{equation*}
$$

This is maximum for forward or backward scattering, and falls to a minimum of half that for perpendicular scattering $\left(\theta_{\mathrm{sc}}=\pi / 2\right)$.

Since the perpendicularly polarized component is scattered with greater effectiveness it will dominate. The scattered light will thus be partially polarized. Its degree of partial polarization is quantified by the fraction

$$
\begin{equation*}
p=\frac{\left\langle S_{r}^{\perp}\right\rangle-\left\langle S_{r}^{\|}\right\rangle}{\left\langle S_{r}^{\perp}\right\rangle+\left\langle S_{r}^{\|}\right\rangle}=\frac{1-\cos ^{2} \theta_{\mathrm{sc}}}{1+\cos ^{2} \theta_{\mathrm{sc}}}=\frac{\sin ^{2} \theta_{\mathrm{sc}}}{2-\sin ^{2} \theta_{\mathrm{sc}}} \tag{64}
\end{equation*}
$$

Perpendicularly scattered light $\left(\theta_{\mathrm{cs}}=\pi / 2\right)$ is fully polarized ( $p=1$ ) since only the perpendicular component is scattered. At other angles there is a mixture of both polarizations, but still dominated by the perpendicular $(0<p<1)$.

## III. Scattering from atoms: Rayleigh scattering

Thomson scattering occurs when a plane wave is incident on a charged free particle. What happens when the wave encounters a neutral particle such as an atom? Rather than delve into the complicated effects the wave's electric field might have on the atom's electron orbitals, we will consider the atom simply as a small polarizable object, like a little dielectric sphere. This is appropriate if the incident wave is not at a frequency which will resonate with the atom's electronic structure.

The polarizable object will develop a dipole moment when immersed in an external electric field, such as that of the incident wave given by eq. (54). We take the atom to be isotropic ${ }^{2}$ with polarizability $\alpha$, so it will develop a dipole moment

$$
\begin{equation*}
\mathbf{p}(t)=\epsilon_{0} \alpha \mathbf{E}(0, t)=\epsilon_{0} \alpha E_{0} \cos (\omega t) \hat{\boldsymbol{\epsilon}}_{0} \tag{65}
\end{equation*}
$$

We have once again placed our object (the atom) at the origin and assumed its dimensions are small compared to the wavelength of the incident wave, $\lambda=2 \pi c / \omega$. Polarizability ${ }^{3}$ has units of $\mathrm{m}^{3}$, and for a uniform sphere of permittivity $\epsilon$ and volume $V$ it will be

$$
\begin{equation*}
\alpha=\frac{3\left(\epsilon-\epsilon_{0}\right)}{\epsilon+2 \epsilon_{0}} V \tag{66}
\end{equation*}
$$

(see eq. [4.56] in Jackson).

[^1]Using eq. (65) in eq. (39) gives a radiated power

$$
\begin{equation*}
P=\frac{\mu_{0} \epsilon_{0}^{2}}{6 \pi c} \alpha^{2} \omega^{4} E_{0}^{2} \cos ^{2}(\omega t)=\frac{\epsilon_{0}}{6 \pi c^{3}} \alpha^{2} \omega^{4} E_{0}^{2} \cos ^{2}(\omega t) . \tag{67}
\end{equation*}
$$

We once again average over a period, and now introduce the wavelength $\lambda=2 \pi c / \omega$. The average power radiated by the dielectric object is

$$
\begin{equation*}
\langle P\rangle=\frac{8 \pi^{3}}{3} \frac{\alpha^{2}}{\lambda^{4}} \epsilon_{0} c \frac{E_{0}^{2}}{2}=\frac{8 \pi^{3}}{3} \frac{\alpha^{2}}{\lambda^{4}} \bar{S}_{0}, \tag{68}
\end{equation*}
$$

where $\bar{S}_{0}=\epsilon_{0} c E_{0}^{2} / 2$ is the mean energy flux of the incident plane wave. This has the same form as expression (57), except this time the scattering cross section is ${ }^{4}$

$$
\begin{equation*}
\sigma_{r}=\frac{8 \pi^{3}}{3} \frac{\alpha^{2}}{\lambda^{4}} \tag{69}
\end{equation*}
$$

This process, called Rayleigh scattering has a cross section that, unlike the Thomson cross section, depends on the wavelength of the incident wave. Its very strong inverse dependence on wavelength, $\sigma_{r} \sim \lambda^{-4}$, is often invoked to explain the blue skies on Earth. Short wavelengths, like blue, are scattered much more effectively by air molecules (see Jackson §10.2.C.).

It is particularly noteworthy that the scattering cross section of our object is not the same as its projected area. For the case of a sphere, eq. (66) shows that $\alpha \sim R^{3} \sim A^{3 / 2}$, where $A=\pi R^{2}$ is the projected area of a sphere. The scattering cross section scales as $\sigma_{r} \sim A^{3} / \lambda^{4}$ rather than being the area $A$. Wavelengths much longer than the sphere's radius, $\lambda \gg A^{1 / 2}$, encounter a cross section far smaller than the sphere would present geometrically. This is why visible light ( $\lambda \simeq 5 \times 10^{-7} \mathrm{~m}$ ) moves so readily through air whose molecules have dimensions $R \sim 3 \times 10^{-10} \mathrm{~m}$. A hypothetical line would hit an air molecule after traveling only $\ell \sim 1 / \pi n R^{2}=10^{-7} \mathrm{~m}-$ an empty bottle is not really so empty! ${ }^{5} \mathrm{~A}$ visible-light photon, however, will travel $(\lambda / R)^{4} \simeq 10^{12}$ times farther: $\ell \simeq 100$ km (more precise values appear in Jackson's $\S 10.2$.C). A photon of much shorter wavelength, say $\lambda=10^{-7} \mathrm{~m}$ from hydrogen's Lyman- $\alpha$ transition, will be stopped $5^{-4} \simeq 10^{-3}$ times sooner, making it only $100 \mathrm{~m}!^{6}$

## The source of energy: Field within the charges

In order to understand the source of the energy lost to radiation we must consider the solutions, (3) and (4), in the opposite limit. We must find the fields inside the charge distribution, $|\mathbf{x}|<a$. These are the fields against which the evolving charges must work, and some of that work must ultimately be carried away as radiation.

To assist our calculation we will consider the time evolution of the charges to be far slower than light travel time across them: $\tau \gg a / c$. This permits a different form of expansion of the retarded

[^2]times. Expanding in powers of $a / c \tau \ll 1$, the charge density in eq. (3) becomes
\[

$$
\begin{align*}
\rho\left(\mathbf{x}^{\prime}, t-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c}\right)= & \rho\left(\mathbf{x}^{\prime}, t\right)-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c} \frac{\partial \rho}{\partial t}+ \\
& +\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^{2}}{2 c^{2}} \frac{\partial^{2} \rho}{\partial t^{2}}-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^{3}}{6 c^{3}} \frac{\partial^{3} \rho}{\partial t^{3}}+\cdots, \tag{70}
\end{align*}
$$
\]

where all derivatives on the right are evaluated at time $t$. The expansion must be carried out to an unusually large number of term because, as we see below, it is necessary to keep the lowest odd order which gives rise to a non-vanishing contribution to the potential.

Placing expansion (70) into the expression for the potential, eq. (3), gives an expansion of the scalar potential

$$
\begin{align*}
\Phi(\mathbf{x}, t)= & \frac{1}{4 \pi \epsilon_{0}}\{\int \frac{\rho\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} d^{3} x^{\prime}-\frac{1}{c} \frac{\partial}{\partial t} \overbrace{\int \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}^{q}+\frac{1}{2 c^{2}} \frac{\partial^{2}}{\partial t^{2}} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right| \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime} \\
& \left.-\frac{1}{6 c^{3}} \frac{\partial^{3}}{\partial t^{3}} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^{2} \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}+\cdots\right\} \\
= & \overbrace{\Phi_{2}(\mathbf{x}, t)}^{\frac{1}{4 \pi \epsilon_{0}} \int \frac{\rho\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} d^{3} x^{\prime}}-0+\overbrace{\frac{\mu_{0}}{8 \pi} \frac{\partial^{2}}{\partial t^{2}} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right| \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}^{\Phi_{0}(\mathbf{x}, t)}+  \tag{71}\\
& \underbrace{-\frac{\mu_{0}}{24 \pi c} \frac{\partial^{3}}{\partial t^{3}} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^{2} \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}_{\Phi_{3}(\mathbf{x}, t)}+\cdots
\end{align*}
$$

$\Phi_{0}(\mathrm{x}, t)$ is the classic electrostatic potential from a static charge distribution (i.e. Jackson's eq. [1.17]), and $\Phi_{2}$ and $\Phi_{3}$, are dynamical corrections to it. The contribution at first order vanishes because, as we have previously noted, charge conservation requires $\dot{q}=0$. Had this remained we could have observed an energetic consequence of monopole radiation. Since this does not occur we need to keep the next two orders to observe the consequences of dipole radiation.

Each of these potential terms contributes differently to the work. To see this consider the rate of work done against this component of the electric field

$$
\begin{equation*}
P_{\Phi}=-\int(-\nabla \Phi) \cdot \mathbf{J}(\mathbf{x}, t) d^{3} x=-\int \Phi(\mathbf{x}, t) \nabla \cdot \mathbf{J} d^{3} x=\int \Phi(\mathbf{x}, t) \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x \tag{72}
\end{equation*}
$$

after integrating by parts and using charge continuity. The contribution of $\Phi_{0}(\mathbf{x}, t)$ to the work is

$$
\begin{align*}
P_{\Phi, 0} & =\int \Phi_{0}(\mathbf{x}, t) \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x=\frac{1}{4 \pi \epsilon_{0}} \int \frac{\rho\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x^{\prime} d^{3} x \\
& =\frac{d}{d t}\left\{\frac{1}{8 \pi \epsilon_{0}} \int \frac{\rho\left(\mathbf{x}^{\prime}, t\right) \rho(\mathbf{x}, t)}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} d^{3} x^{\prime} d^{3} x\right\}, \tag{73}
\end{align*}
$$

after making us of the symmetry between $\mathbf{x}$ and $\mathbf{x}^{\prime}$ in the integral. This shows that the electric field generated by $\Phi_{0}(\mathbf{x}, t)$ contributes conservatively to the work. There is an energy

$$
\begin{equation*}
W_{\Phi, 0}=\frac{1}{8 \pi \epsilon_{0}} \int \frac{\rho\left(\mathbf{x}^{\prime}, t\right) \rho(\mathbf{x}, t)}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} d^{3} x^{\prime} d^{3} x \tag{74}
\end{equation*}
$$

which accounts for the work being done. This is the classic electrostatic self-energy of the charge distribution, given in Jackson as eq. (1.52).

A similar process occurs for $\Phi_{2}(\mathbf{x}, t)$

$$
\begin{align*}
P_{\Phi, 2} & =\frac{\mu_{0}}{8 \pi} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right| \frac{\partial^{2} \rho\left(\mathbf{x}^{\prime}, t\right)}{\partial t^{2}} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x^{\prime} d^{3} x \\
& =\frac{d}{d t}\left\{\frac{\mu_{0}}{16 \pi} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right| \frac{\partial \rho\left(\mathbf{x}^{\prime}, t\right)}{\partial t} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x^{\prime} d^{3} x\right\}, \tag{75}
\end{align*}
$$

where we once again make use of the obvious symmetry in the double integral. In this case the conserved energy is

$$
\begin{align*}
W_{\Phi, 2} & =\frac{\mu_{0}}{16 \pi} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right| \frac{\partial \rho\left(\mathbf{x}^{\prime}, t\right)}{\partial t} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^{3} x^{\prime} d^{3} x \\
& =\frac{\mu_{0}}{16 \pi} \int\left|\mathbf{x}-\mathbf{x}^{\prime}\right|\left(\nabla^{\prime} \cdot \mathbf{J}\right)(\nabla \cdot \mathbf{J}) d^{3} x^{\prime} d^{3} x . \tag{76}
\end{align*}
$$

While not a familiar expression, this is a dynamical form of energy which will increase or decrease as charges are moved. If we begin and end our process with a static distribution of charges and currents then $\nabla \cdot \mathbf{J}=0$, and $W_{\Phi, 2}$ will be zero at the beginning and at the end. This term will have contributed no net work in that case. While energy is certainly lost to radiation, this term does not reflect that loss.

This pattern appears ready to repeat at all higher orders which arise from even time derivatives in eq. (71). The product with a first derivative, $\partial \rho / \partial t$, can be re-written as total time derivative minus another product of even and odd time derivatives. This repeats up to a total time derivative of a symmetric double integral. The result is a sum of total time derivatives which constitutes a conservative contribution to the work.

Because $\Phi_{3}(\mathbf{x}, t)$ involves an odd number of time derivatives, its contribution to the work will not be a total time derivative as the previous two orders have been. It will give rise to work which is not conservative. Instead of calculating it, we consider the electric field generated by this potential

$$
\begin{align*}
-\nabla \Phi_{3} & =\frac{\mu_{0}}{12 \pi c} \frac{\partial^{3}}{\partial t^{3}} \int\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime} \\
& =\frac{\mu_{0}}{12 \pi c} \frac{\partial^{3}}{\partial t^{3}}[\mathbf{x} \underbrace{\int \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}_{q(t)}-\underbrace{\int \mathbf{x}^{\prime} \rho\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}_{\mathbf{p}(t)}]=-\frac{\mu_{0}}{12 \pi c} \dot{\mathbf{p}}(t) . \tag{77}
\end{align*}
$$

It seems that this contribution yields a uniform electric field throughout the charged body.
To complete the calculation of work we must compute the inductive contribution to the electric field. To do this we compute the vector potential using a similar expansion in powers of $a / c \tau \ll 1$

$$
\mathbf{J}\left(\mathbf{x}^{\prime}, t-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c}\right)=\mathbf{J}\left(\mathbf{x}^{\prime}, t\right)-\frac{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|}{c} \frac{\partial \mathbf{J}}{\partial t}+\cdots,
$$

although not to as many orders, since the first order term will give rise to a non-vanishing contribution. Using this in eq. (4) yields a vector potential

$$
\mathbf{A}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi} \int \frac{\mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime}-\frac{\mu_{0}}{4 \pi c} \frac{\partial}{\partial t} \overbrace{\int \mathbf{J}\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}}^{\dot{\mathbf{p}}(t)}+\cdots
$$

$$
\begin{equation*}
=\underbrace{\frac{\mu_{0}}{4 \pi} \int \frac{\mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime}}_{\mathbf{A}_{0}(\mathbf{x}, t)} \underbrace{-\frac{\mu_{0}}{4 \pi c} \ddot{\mathbf{p}}(t)}_{\mathbf{A}_{u}(t)}+\cdots \tag{78}
\end{equation*}
$$

after once again making use of eq. (25). The first term is the classic magnetostatic vector potential. Its contribution to work

$$
\begin{align*}
P_{i, 0} & =-\int\left(-\frac{\partial \mathbf{A}_{0}}{\partial t}\right) \cdot \mathbf{J}(\mathbf{x}, t) d^{3} x=\frac{\mu_{0}}{4 \pi} \int \frac{\partial \mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\partial t} \cdot \frac{\mathbf{J}(\mathbf{x}, t)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime} d^{3} x \\
& =\frac{d}{d t}\left\{\frac{\mu_{0}}{8 \pi} \int \frac{\mathbf{J}(\mathbf{x}, t) \cdot \mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime} d^{3} x\right\} \tag{79}
\end{align*}
$$

is also conservative. The conserved energy

$$
\begin{equation*}
W_{i, 0}=\frac{\mu_{0}}{8 \pi} \int \frac{\mathbf{J}(\mathbf{x}, t) \cdot \mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime} d^{3} x \tag{80}
\end{equation*}
$$

is the classic magnetostatic energy (see eq. [5.153] in Jackson).
Moving beyond this conservative term we find a uniform inductive electric field from the uniform vector potential $\mathbf{A}_{u}(t)$. Combing this with the uniform electrostatic field given in (77),

$$
\begin{equation*}
\mathbf{E}_{u}=-\nabla \Phi_{3}-\frac{\partial \mathbf{A}_{u}}{\partial t}=-\frac{\mu_{0}}{12 \pi c} \dot{\ddot{\mathbf{p}}}(t)+\frac{\mu_{0}}{4 \pi c} \dot{\mathbf{p}}(t)=\frac{\mu_{0}}{6 \pi c} \dot{\ddot{\mathbf{p}}}(t) \tag{81}
\end{equation*}
$$

gives a uniform field throughout the charge distribution. This uniform electric field contributes to the work

$$
\begin{align*}
P_{u} & =-\int \mathbf{E}_{u}(t) \cdot \mathbf{J}(\mathbf{x}, t) d^{3} x=-\frac{\mu_{0}}{6 \pi c} \ddot{\ddot{\mathbf{p}}} \cdot \overbrace{\int \mathbf{J}(\mathbf{x}, t) d^{3} x}^{\dot{\mathbf{p}}(t)}=-\frac{\mu_{0}}{6 \pi c} \ddot{\ddot{\mathbf{p}}} \cdot \dot{\mathbf{p}} \\
& =\frac{d}{d t}\left(-\frac{\mu_{0}}{6 \pi c} \ddot{\mathbf{p}} \cdot \dot{\mathbf{p}}\right)+\frac{\mu_{0}}{6 \pi c}|\ddot{\mathbf{p}}|^{2}=\frac{d^{2}}{d t^{2}}\left(-\frac{\mu_{0}}{12 \pi c}|\dot{\mathbf{p}}|^{2}\right)+\frac{\mu_{0}}{6 \pi c}|\ddot{\mathbf{p}}|^{2}, \tag{82}
\end{align*}
$$

after making use of eq. (25). The second term in the final expression is positive work done work lost - against the uniform electric field. Notably, it exactly matches the power carried away by dipole radiation, given by expression (39). This correspondence between the uniform internal electric field and the dipolar external contribution is rather natural, since both arise from $\ell=1$ terms in spherical harmonic expansion of the potential.

This derivation has shown that much of the work done moving charges around is stored in the reconfigured electric and magnetic fields. This work is conserved and will be recovered if the charges are moved back again. There is one contribution, however, that is entirely lost and cannot be recovered. This is the energy that is ultimately carried away by dipole radiation. Had we carried expansions to still higher order in $a / c \tau \ll 1$ we would have recovered still more conserved energies. As argued above, terms of even order will lead to only such conserved contributions. The higher odd orders, with increasing power of $c$ in their denominators, will include energy losses as well. These will be found to be losses from other radiation orders: magnetic dipole, electric quadrupole, etc. We have stopped our derivation at the order that corresponds to losses from dipole radiation alone.

## Radiation reaction

It is simplest to apply the above result to the case of a single point charge following trajectory $\mathbf{r}(t)$. The dipole moment of this lone charge is $\mathbf{p}(t)=q \mathbf{r}(t)$. Its motion creates a uniform electric field

$$
\begin{equation*}
\mathbf{E}_{\mathrm{rad}}(t)=\frac{\mu_{0} q}{6 \pi c} \dot{\mathbf{r}}(t) \tag{83}
\end{equation*}
$$

according to eq. (81). The charge naturally gives rise to a singular electrostatic field, $\sim r^{-2}$, but this does not affect it - it points in all directions away from the charge. In contrast to this, the electric field in eq. (83), is uniform at the charge, and the charge does feel it. It creates a force on the charge

$$
\begin{equation*}
\mathbf{F}_{\mathrm{rad}}=q \mathbf{E}_{\mathrm{rad}}=\frac{\mu_{0} q^{2}}{6 \pi c} \dot{\mathbf{r}}(t) \tag{84}
\end{equation*}
$$

known as the radiation reaction or the Abraham-Lorentz force. ${ }^{7}$ It famously leads to unphysical results when used in the dynamical evolution of a free charge. Our original application was to find the field against which a prescribed charge evolution must work. This does not suffer from the same drawbacks.

## Radiation resistance

Current in the dipole antenna, for example, is driven by some external circuit. The circuit must work against an electric field which is approximately uniform in the vicinity of the antenna itself. Using equation (47) in eq. (81) gives that electric field

$$
\begin{equation*}
\mathbf{E}_{u}=\frac{\mu_{0}}{6 \pi c} \ell_{\mathrm{eff}} \dot{\tilde{q}}_{a} \hat{\mathbf{\imath}}=\frac{\mu_{0}}{6 \pi c} \ell_{\mathrm{eff}} \ddot{I}_{a}(t) \hat{\mathbf{l}} . \tag{85}
\end{equation*}
$$

This must be contrasted with the familiar "back EMF", proportional to a self-inductance and the first derivative of current, $\sim \mu_{0} \ell_{\text {eff }} \dot{I}_{a}$. That contribution is found from Faraday's law alone, and work done against it is stored in the magnetostatic field (i.e. the energy is conserved). In fact, the self energy $W_{i, 0}$, given by eq. (80), and found from the magnetostatc contribution, $\mathbf{A}_{0}$, is often used to compute the self-inductance of a circuit.

The electric field given in eq. (85) is thus different from that derived from self-inductance. It must arise from both Faraday's law and the displacement current. That combination gives rise to the factor $c$ in its denominator. It is a genuinely radiative effect. Were we to integrate the electric field across the antenna we would obtain a "voltage"

$$
\begin{equation*}
V_{\mathrm{rad}} \sim-E_{u} \ell_{\mathrm{eff}}=-\frac{\mu_{0} \ell_{\mathrm{eff}}^{2}}{6 \pi c} \ddot{I}_{a} \tag{86}
\end{equation*}
$$

It is this back-voltage that the circuit must fight and requires power $I_{a} V_{\mathrm{rad}}$ to do so. In our derivation we have specified the current, $I_{a}(t)$, so the driving circuit is presumably able to supply this work. Here is where we avoid the difficulties encountered when using the Abraham-Lorentz force in self-consistent dynamics. Notably the power supplied by the circuit exactly matches the power lost to radiation given in eq. (50) - albeit in reworked terms.

[^3]Finally, we note that the term "radiation resistance" is sometimes used to describe this effect. That can be misleading ${ }^{8}$ since a genuine resistance would produce a voltage $V=\mathcal{R} I_{a}$, instead of the expression given by (86), proportional to $\ddot{I}_{a}$. Assuming a purely sinusoidal current allows us to side-step this profound difference and write a resistance

$$
\begin{equation*}
\mathcal{R}_{\mathrm{rad}}=\frac{\mu_{0} \ell_{\mathrm{eff}}^{2} \omega^{2}}{6 \pi c}=\frac{2 \pi}{3} c \mu_{0}\left(\frac{\ell_{\mathrm{eff}}}{\lambda}\right)^{2}=Z_{r}\left(\frac{\ell_{\mathrm{eff}}}{\lambda}\right)^{2}, \tag{87}
\end{equation*}
$$

after replacing $\omega=2 \pi c / \lambda$ to match the notation in eq. (50).
Comparing this to the impedance from the circuit's actual self-inductance, $\mathcal{L}=\mu_{0} \ell_{\text {eff }}$, yields the ratio

$$
\begin{equation*}
\frac{\mathcal{R}_{\mathrm{rad}}}{\left|Z_{\mathcal{L}}\right|}=\frac{\mathcal{R}_{\mathrm{rad}}}{\omega \mu_{0} \ell_{\mathrm{eff}}}=\frac{1}{6 \pi} \frac{\ell_{\mathrm{eff}} \omega}{c}=\frac{1}{3} \frac{\ell_{\mathrm{eff}}}{\lambda} . \tag{88}
\end{equation*}
$$

Radiation is therefore a very small effect, compared to self-inductance, until a circuit is operating at frequencies comparable to light transit across itself. For table-sized circuits, this means frequencies in the gigahertz range or above. At lower frequencies the radiation resistance will play only a minor role in the circuit. This justifies its neglect and explains why you have probably never encountered radiation resistance in circuit analysis. The statement is equivalent to the conclusion that the displacement current, which was required to produce the effect, plays only a minor role at frequencies below the light-travel time. It does lead to a loss which, however, small, is different from the effect of self-inductance. This loss is not dissipated as heat, as for genuine resistance, but is carried away as radiation.

## How do we get a uniform field?

The foregoing has shown that formal solutions to the full set of Maxwell's equations, eqs. (3) and (4), include a uniform electric field proportional to the third time-derivative of the total electric dipole. This is somewhat different from solutions we typically obtain from static analysis. How does it arise? We show here how we can obtain this solution (approximately) following the standard steps. It is the result of combining Faraday's law of induction with the displacement current in Ampère's law, and matching to a propagating wave solution far away.

We begin by solving Ampère's law for a magnetic field from the distribution of currents at some time. In this initial step we omit the displacement current and attempt to solve the familiar equation

$$
\begin{equation*}
\nabla \times \mathbf{B}_{0}=\mu_{0} \mathbf{J}(\mathbf{x}, t) \tag{89}
\end{equation*}
$$

It is not possible to solve this exactly since $\nabla \cdot \mathbf{J} \neq 0$, but an approximate solution ${ }^{9}$ is $\mathbf{B}_{0}=\nabla \times \mathbf{A}_{0}$, where the vector potential is the first term in eq. (78). We next use this solution in Faraday's law

[^4]to find an inductive electric field
\[

$$
\begin{equation*}
\nabla \times \mathbf{E}_{i}=-\frac{\partial \mathbf{B}_{0}}{\partial t}=\nabla \times\left(-\frac{\partial \mathbf{A}_{0}}{\partial t}\right) \tag{90}
\end{equation*}
$$

\]

The most general solution to this equation can be written in the form

$$
\begin{equation*}
\mathbf{E}_{i}(\mathbf{x}, t)=-\frac{\partial \mathbf{A}_{0}(\mathbf{x}, t)}{\partial t}+\nabla \Psi+\mathbf{E}_{i, h}(\mathbf{x}, t) \tag{91}
\end{equation*}
$$

where $\mathbf{E}_{i, h}(\mathbf{x}, t)$ is the homogeneous contribution with no divergence and no curl

$$
\begin{equation*}
\nabla \cdot \mathbf{E}_{i, h}=0 \quad, \quad \nabla \times \mathbf{E}_{i, h}=0 \tag{92}
\end{equation*}
$$

In ordinary analysis we choose this function to satisfy boundary conditions. Our problem has no actual boundaries, so we will need to be a bit clever about how we find our homogeneous contribution. The scalar function $\Psi(\mathbf{x}, t)$ takes the place of choosing a specific gauge for $\mathbf{A}_{0}(\mathbf{x}, t)$. We choose it so that the inductive electric field is entirely divergence-free

$$
\begin{equation*}
\nabla \cdot \mathbf{E}_{i}=-\nabla \cdot\left(\frac{\partial \mathbf{A}_{0}(\mathbf{x}, t)}{\partial t}\right)+\nabla^{2} \Psi+\underbrace{\nabla \cdot \mathbf{E}_{i, h}}_{=0}=0 \tag{93}
\end{equation*}
$$

We thus require $\nabla^{2} \Psi=\nabla \cdot\left(\partial \mathbf{A}_{0} / \partial t\right)$ and also require that $|\nabla \Psi| \rightarrow 0$ as $r \rightarrow \infty$.
Considering now the region lying well outside the charges, $r \gg a$, but inside the static region, $r \ll c \tau$, we can expand the vector potential $\mathbf{A}_{0}$ in a traditional multipole expansion

$$
\begin{equation*}
\mathbf{A}_{0}(\mathbf{x}, t)=\frac{\mu_{0}}{4 \pi} \int \frac{\mathbf{J}\left(\mathbf{x}^{\prime}, t\right)}{\left|\mathbf{x}^{\prime}-\mathbf{x}\right|} d^{3} x^{\prime} \simeq \frac{\mu_{0}}{4 \pi} \frac{1}{r} \int \mathbf{J}\left(\mathbf{x}^{\prime}, t\right) d^{3} x^{\prime}+\cdots=\frac{\mu_{0}}{4 \pi} \frac{\dot{\mathbf{p}}(t)}{r}+\cdots \tag{94}
\end{equation*}
$$

where have once more used eq. (25). The divergence of the asymptotic version,

$$
\nabla \cdot\left(\frac{\partial \mathbf{A}_{0}}{\partial t}\right)=\frac{\mu_{0}}{4 \pi} \nabla \cdot\left(\frac{\ddot{\mathbf{p}}}{r}\right)=-\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^{2}},
$$

demands a scalar function whose asymptotic form is

$$
\begin{equation*}
\Psi=\frac{\mu_{0}}{8 \pi} \ddot{\mathbf{p}} \cdot \hat{\mathbf{r}} \quad \Longrightarrow \quad \nabla \Psi=\frac{\mu_{0}}{8 \pi}\left[\frac{\ddot{\mathbf{p}}}{r}-\frac{(\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}}{r}\right] \tag{95}
\end{equation*}
$$

While the potential does not vanish as $r \rightarrow \infty$, its gradient does. It will therefore be the role of the homogeneous component, $\mathbf{E}_{i, h}$, to account for any "boundary" conditions. The full inductive field will have an asymptotic form

$$
\begin{equation*}
\mathbf{E}_{i}(\mathbf{x}, t) \simeq-\frac{\mu_{0}}{8 \pi}\left[\frac{\ddot{\mathbf{p}}}{r}+\frac{(\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}}{r}\right]+\mathbf{E}_{i, h}(\mathbf{x}, t) \tag{96}
\end{equation*}
$$

This does not fall off as a monopole because its source, $-\partial \mathbf{B}_{0} / \partial t$, is not localized to $r<a$ even though the currents and charges are.

It is at this stage that we finally introduce the displacements current. The original magnetic field, $\mathbf{B}_{0}(\mathbf{x}, t)$, satisfied Ampère's law without it. To this we add a magnetic field, $\mathbf{B}_{a d f}(\mathbf{x}, t)$, which accounts only for the displacement current in Ampère's law

$$
\begin{equation*}
\nabla \times \mathbf{B}_{a d f}=\mu_{0} \epsilon_{0} \frac{\partial \mathbf{E}_{i}}{\partial t}=-\frac{\mu_{0}}{8 \pi c^{2}}\left[\frac{\dot{\mathbf{p}}}{r}+\frac{(\dot{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}}{r}\right]+\frac{1}{c^{2}} \frac{\partial \mathbf{E}_{i, h}}{\partial t} \tag{97}
\end{equation*}
$$

where the final expression applies in the region $a \ll r \ll c \tau$ (the sub-script is a reminder that this magnetic field arises from applying Ampre's law to the Displacement current from the electric field which solved Faraday's law). We show below that the final term is small in our scaling. Ignoring it we are able to find a solution

$$
\begin{equation*}
\mathbf{B}_{a d f}(\mathbf{x}, t)=\frac{\mu_{0}}{8 \pi c^{2}} \hat{\mathbf{r}} \times \dot{\hat{\mathbf{p}}}(t) \tag{98}
\end{equation*}
$$

which can be easily checked. This solution fails to fall off at all with distance owing to the fact that its source, the displacement current, falls of only as $1 / r$.

We begin to sense that the combination of Faraday's law and the displacement current will lead to solutions which grow rather than decay with distance. This occurs because the combination will admit traveling wave solutions, even in the absence of sources. To proceed with some degree of selfconsistency we attempt to make these asymptotic fields match the form of an electromagnetic wave. To do this we seek an homogeneous inductive electric field that, with the asymptotic magnetic field, appears like an outward traveling wave

$$
\begin{equation*}
\mathbf{B}_{a d f} \simeq \frac{1}{c} \hat{\mathbf{r}} \times \mathbf{E}_{i, h} \quad, \quad r \rightarrow \infty \tag{99}
\end{equation*}
$$

The homogeneous inductive field is thus

$$
\begin{equation*}
\mathbf{E}_{i, h}(\mathbf{x}, t)=\frac{\mu_{0}}{8 \pi c} \dot{\hat{\mathbf{p}}}(t) \quad, \quad r \rightarrow \infty \tag{100}
\end{equation*}
$$

asymptotically. We are seeking a field for smaller $r$ which is curl-free and divergence-free (see eq. [92]) and matches the constant vector eq. (100) as $r$ gets large. The unique solution to this problem is the constant vector itself: eq. (100). In other words, this is the homogeneous solution, at least to this order, and applies throughout the static zone and even within the charges - it applies for all $r \ll c \tau$.

We can return our version of $\mathbf{E}_{i, h}$ to eq. (97) to obtain a correction to $\mathbf{B}_{a d f}(\mathbf{x}, t)$. This correction will scale as $r$ - it actually grows with distance - and will involve a fourth time-derivative of $\mathbf{p}(t)$. Its magnitude will be $\sim(r / c \tau) \mathbf{B}_{a d f}$, compared to eq. (98). Since we are working within the static zone, $r \ll c \tau$, the correction will be a small one.

The form of $\mathbf{E}_{h, i}$, given in eq. (100) matches that of (81), although it is small by a factor of 3/4. That discrepancy must be the result of our ad hoc attempt at asymptotic matching after only a single iteration through Faraday's and Ampère's laws. Among the contributions we did not consider is the magnetic field arising from the displacement current of the changing electrostatic field. This displacement current, $\sim \mu_{0} \dot{p} / r^{3}$, will create a magnetic field $B_{a d s} \sim \mu_{0} \dot{p} / r^{2}$. The inductive electric field from this, $E \sim \mu_{0} \ddot{p} / r$, scales the same as that in eq. (96). Its contribution should thus have been considered in computing $\mathbf{E}_{i, h}$. Including this, as well all the other contributions, is not necessary since the derivation of eq. (81) include all terms: it worked from eqs. (3) and (4),the exact solutions to Maxwell's equations. This latest exercise was simply intended to provide insight into how the different pieces of Maxwell's equations, particularly induction and displacement current, produce a uniform electric field $\sim \dot{\overline{\mathbf{p}}}$.


[^0]:    ${ }^{1}$ This very important discussion is not given in Jackson until $\S 9.1$. It is conceptually very significant, and does not require all the formalism developed in those later chapters.

[^1]:    ${ }^{2}$ It is not difficult to repeat this with a tensor polarizability. This has very interesting effects on the polarization of the scattered radiation.
    ${ }^{3}$ See eg. eq. (4.67) in Jackson, where polarizability is denoted $\gamma$ instead of $\alpha$.

[^2]:    ${ }^{4}$ This matches Jackson's eq. (10.11) after replacing $\alpha$ with expression (66), and setting $V=4 \pi a^{3} / 3$.
    ${ }^{5}$ This is essentially the mean-free path of an air molecule. It is extremely small, which is why air behaves like a continuous fluid rather than a collection of particles.
    ${ }^{6}$ This simple picture does not account for the interactions of such photons with the electronic structure of the atom. Doing so makes the trip shorter still.

[^3]:    ${ }^{7}$ This is given by Jackson as eq. (16.8), although in cgs units. To obtain that form from SI simply replace $q^{2} \rightarrow 4 \pi \epsilon_{0} e^{2}=4 \pi e^{2} / \mu_{0} c^{2}$.

[^4]:    ${ }^{8}$ The term actually refers to nature of the impedance which is found from complex current $I(t) \sim e^{-i \omega t}$. The second derivative brings down a factor $(-i \omega)^{2}=-\omega^{2}$, leading to a purely real impedance. Real impedances are called "resistive" to distinguish them form purely imaginary, "reactive", impedances arising from odd derivatives of current, as in inductors or capacitors. (The latter involves a single anti-derivative of current bringing down a factor of $i / \omega)$. A real (resistive) impedance leads to a voltage whose phase either matches the current or is $180^{\circ}$ from it. This leads to net power when averaged over a period. Imaginary (reactive) impedances, on the other hand, produce voltages $\pm 90^{\circ}$ out of phase with the current, and the power integrates to zero over a period. Impedance thus offers one more bit of insight into the need for odd terms in expansions of the potential like (71): odd derivatives of $\rho$ are even derivatives of $\mathbf{J}$, and thus lead to real (resistive) impedances, and therefore to losses.
    ${ }^{9}$ We would be more correct at this order if we also included the displacement current from the electrostatic field $\mu_{0} \epsilon_{0} \dot{E}_{s} \sim \mu_{0} \dot{p} / r^{3}$. We discuss below how this more complicated contribution works in parallel with what we do here. It modifies the numerical factor and provides no further insight.

