

Capacitance of a sphere near a ground plane — images charges to infinity

(Dana Longcope, 1/17/2019)

It is possible to exactly write the capacitance of a conducting sphere of radius a : $C = 4\pi\epsilon_0 a$. Doing so requires the sphere to be isolated in the universe, so its potential may fall as $\Phi \sim 1/r$ in every direction. The presence of the ground, or some grounded plane, anywhere around the sphere will violate this condition, and affect the capacitance. It is possible to find an expression for the capacitance in this case, although only with some effort.

To perform the calculation we locate sphere of radius a with its center at the origin, and take the horizontal ground plane *above* it at $z = h$. To compute capacitance we set $\Phi = 1$ on the conductor, $r = a$, maintain ground ($z = h$) at $\Phi = 0$, and then determine the charge on the sphere. We perform this computation with an infinite series of image charges, q_0, q_1, q_2, \dots located at positions z_0, z_1, z_2, \dots within the sphere ($z_n < a$). Each of these has an oppositely charged mirror image $q'_0 = -q_0, q'_1 = -q_1, \dots$ located at positions z'_0, z'_1, z'_2, \dots , above the ground plane ($z'_n > h$). This could prove to be a formidable task, but it is repetitive and can, in the end, be solved exactly.

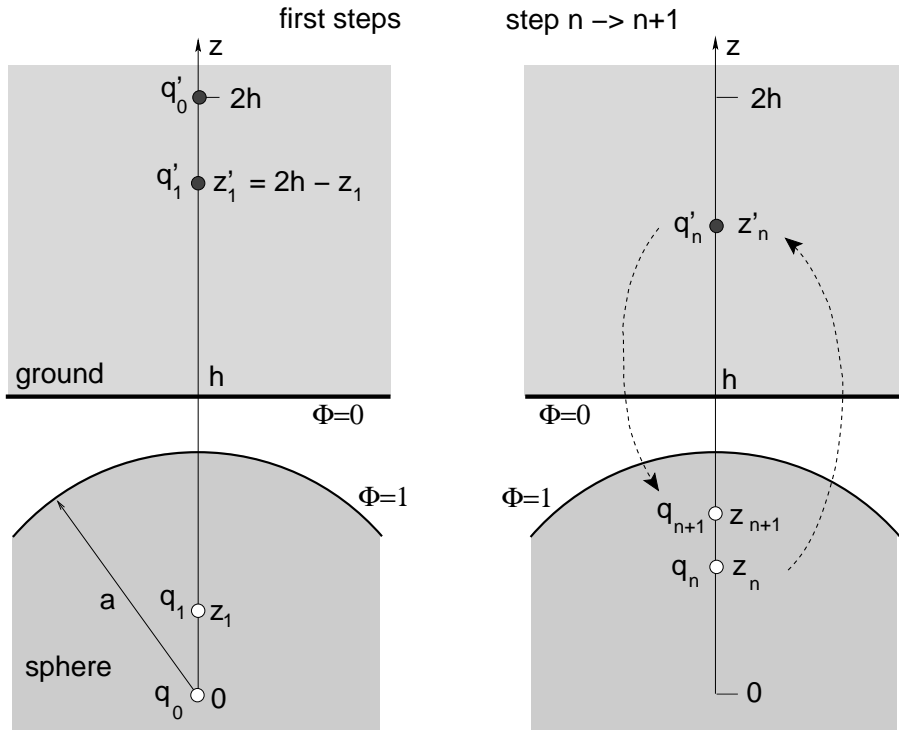


Fig. 1.— The computation of the capacitance using a series of image charges. The left panel shows the placement of the first four charges, $q_0, q'_0, q_1,$ and q'_1 , at positions $z_0 = 0, z'_0 = 2h, z_1,$ and z'_1 . Positive and negative charges are denoted by white and black circles. The right panel shows the general process, $q_n \rightarrow q'_n \rightarrow q_{n+1}$, using dashed arrows.

We build our sets of image charges iteratively as illustrated in Fig. 1. Unprimed variables

denote charges inside the sphere and primes denote those within the ground ($z'_n > h$). We begin by setting $\Phi(r = a) = 1$ using the charge

$$q_0 = 4\pi\epsilon_0 a \quad , \quad (1)$$

placed at the origin: $z_0 = 0$. While this achieves the first objective, of making $\Phi(r = a) = 1$, it creates a non-zero potential at the ground plane. Our next step is to restore that condition using the mirror image, $q'_0 = -q_0$, located at

$$z'_0 = 2h \quad , \quad (2)$$

above the ground plane, as shown on the left of Fig. 1.

The image charge q'_0 changes the potential at $r = a$, and must therefore be matched by another image charge inside the conductor. This will be located at

$$z_1 = \frac{a^2}{z'_0} = \frac{a^2}{2h} \quad , \quad (3)$$

and have charge

$$q_1 = -\frac{a}{z'_0} q'_0 = \frac{a}{2h} q_0 \quad , \quad (4)$$

according to eq. (2.4) of Jackson.

This new charge has, however, disrupted the condition $\Phi(z = h) = 0$. Restoring that condition requires the new image be matched by its own image, $q'_1 = -q_1$, located at

$$z'_1 = 2h - z_1 \quad . \quad (5)$$

The pattern is clear: when we add an image charge in order to satisfy one condition, it breaks the other condition. Clearly this process will never end. We might hope, however, that successive generations of image charges grow smaller, and there is ultimately some convergence. We show below that it does converge, and it is possible to obtain a result to arbitrary order quite readily.

To generalize the process we start at a stage where we have just placed a charge q_n at $z_n < a$, inside the conductor, and thereby achieved $\Phi(r = a) = 1$. As we saw above, the condition $\Phi(z = h) = 0$ requires that this latest charge be balanced by its mirror image, $q'_n = -q_n$, located above the ground plane at

$$z'_n = 2h - z_n \quad ; \quad (6)$$

the first step illustrated on the right of Fig. 1. This addition disrupts the $\Phi(r = a) = 1$ condition. Restoring it requires another image charge inside the sphere at

$$z_{n+1} = \frac{a^2}{z'_n} = \frac{a^2}{2h - z_n} \quad , \quad (7)$$

illustrated on the right of Fig. 1. This image will have charge

$$q_{n+1} = -\frac{a}{z'_n} q'_n = \frac{a}{z'_n} q_n = \frac{z_{n+1}}{a} q_n \quad , \quad (8)$$

where the final step has used eq. (7) to replace $z'_n = a^2/z_{n+1}$.

By eliminating z'_n we obtain recursion relations, (7) and (8), giving the location and charge of the next enclosed image charge (i.e. unprimed) explicitly from the last. Only the enclosed image charges contribute to the net charge of the conductor and are the only ones we need to compute the capacitance.

Recursion relations (7) and (8) are initiated with

$$q_0 = 4\pi\epsilon_0 a \quad , \quad z_0 = 0 \quad . \quad (9)$$

For $n = 0$, where $z_n = 0$, eq. (7) returns the value of z_1 given in eq. (3). Placing this into the $n = 0$ version of eq. (8) returns charge q_1 matching (4). We can then use these values in (7) and (8), with $n = 1$, to obtain

$$z_2 = \frac{a^2}{2h - a^2/2h} = a \frac{a/2h}{1 - (a/2h)^2} \quad , \quad (10)$$

and

$$q_2 = \frac{z_2}{a} q_1 = \frac{a/2h}{1 - (a/2h)^2} q_1 = \frac{(a/2h)^2}{1 - (a/2h)^2} q_0 \quad , \quad (11)$$

respectively. It is evident here that $q_1 \sim (a/h) q_0$, and $q_2 \sim (a/h)^2 q_0$. This scaling will continue, so we can hope that the successive image charges are weaker, especially if $a \ll h$.

It turns out that a rather obscure trick allows us to find a closed form expression for image location z_n and charge q_n . The trick is to introduce a new variable ξ to replace h :

$$h = a \cosh(\xi) \quad , \quad (12)$$

which is possible because $h > a$. In terms of this new variable we can (with some work) obtain the closed form expression

$$z_n = a \frac{\sinh(n\xi)}{\sinh[(n+1)\xi]} \quad . \quad (13)$$

The origin of this expression may appear mysterious. All that matters is that it satisfies the recursion relation, eq. (7), and the initial condition $z_0 = 0$. Provided it does both things, it must be the solution regardless of how it was obtained (to satisfy the curious, I show at the end how I obtained it).

Setting $n = 0$ in eq. (13) confirms that the proposed solution agrees with the initial condition is $z_0 = 0$. To verify that it also satisfies the recursion relation, the expression is inserted into the right hand side of eq. (7)

$$\begin{aligned} z_{n+1} &= \frac{a^2}{2a \cosh \xi - z_n} = \frac{a}{2 \cosh \xi - \sinh(n\xi)/\sinh[(n+1)\xi]} \\ &= \frac{a \sinh[(n+1)\xi]}{2 \sinh[(n+1)\xi] \cosh \xi - \sinh(n\xi)} \quad . \end{aligned} \quad (14)$$

We then use the hyperbolic identity

$$2 \sinh[(n+1)\xi] \cosh \xi = \sinh[(n+2)\xi] + \sinh(n\xi) , \quad (15)$$

in the denominator to obtain

$$z_{n+1} = a \frac{\sinh[(n+1)\xi]}{\sinh[(n+2)\xi]} . \quad (16)$$

This is another version of the proposed solution (13), demonstrating that expression (13) satisfies the recursion relation (7).

Expression (13) can be used to write a closed form solution for charge q_n . We first iterate relation (8)

$$q_n = \frac{z_n}{a} q_{n-1} = \left(\frac{z_n}{a}\right) \left(\frac{z_{n-1}}{a}\right) q_{n-2} = \left(\frac{z_n}{a}\right) \left(\frac{z_{n-1}}{a}\right) \cdots \left(\frac{z_j}{a}\right) \cdots \left(\frac{z_1}{a}\right) q_0 . \quad (17)$$

We use (13) to write each of the factors in parentheses as a ratio

$$\frac{z_j}{a} = \frac{\sinh(j\xi)}{\sinh[(j+1)\xi]} .$$

This has the property its numerator will be canceled by the denominator of the next factor

$$\frac{z_{j-1}}{a} = \frac{\sinh[(j-1)\xi]}{\sinh(j\xi)} .$$

This happens for every factor but the last, leaving only the denominator of the first and the numerator of the last

$$q_n = \frac{\sinh(\xi)}{\sinh[(n+1)\xi]} q_0 , \quad (18)$$

Expression (18). naturally returns $q_0 = q_0$ for $n = 0$. For $n = 1$ it yields

$$q_1 = \frac{\sinh(\xi)}{\sinh(2\xi)} q_0 = \frac{\sinh(\xi)}{2 \sinh(\xi) \cosh(\xi)} q_0 = \frac{1}{2 \cosh(\xi)} q_0 = \frac{a}{2h} q_0 , \quad (19)$$

after replacing $\cosh \xi = h/a$, per its definition (12). This matches the first image charge we found in eq. (4). For $n = 2$ version we require the slightly more obscure identity

$$\sinh(3\xi) = \sinh(\xi) [4 \cosh^2(\xi) - 1] , \quad (20)$$

to obtain

$$q_2 = \frac{\sinh(\xi)}{\sinh(3\xi)} q_0 = \frac{q_0}{4 \cosh^2(\xi) - 1} = \frac{q_0}{(2h/a)^2 - 1} = \frac{(a/2h)^2}{1 - (a/2h)^2} q_0 , \quad (21)$$

exactly as in eq. (11).

It is now easy to obtain the full capacitance by adding up the infinity of image charges inside the sphere

$$C = Q = \sum_{n=0}^{\infty} q_n = q_0 \sinh(\xi) \sum_{n=0}^{\infty} \frac{1}{\sinh[(n+1)\xi]} . \quad (22)$$

To finish the derivation we use eq. (1) to replace q_0 and obtain a version

$$C = 4\pi\epsilon_0 a F [\cosh^{-1}(h/a)] , \quad (23)$$

where the dimensionless function is

$$F(\xi) = \sinh(\xi) \sum_{n=0}^{\infty} \frac{1}{\sinh[(n+1)\xi]} . \quad (24)$$

Properties of the capacitance

The capacitance is defined by the function $F(\xi)$ given by the series in eq. (24). This series comes from the summation of all of the image charges, and it can be shown to converge for every value of $\xi > 0$, meaning every height $h > a$. The ratio of successive terms is the same as the ratio of successive charges

$$\frac{q_{n+1}}{q_n} = \frac{\sinh[(n+1)\xi]}{\sinh[(n+2)\xi]} . \quad (25)$$

Late in the series, where n gets large enough that $n\xi \gg 1$, the ratio becomes

$$\frac{\sinh[(n+1)\xi]}{\sinh[(n+2)\xi]} \simeq \frac{e^{(n+1)\xi}}{e^{(n+2)\xi}} = e^{-\xi} . \quad (26)$$

As long as $\xi > 0$ (i.e. $h > a$) this ratio is less than one, so the series will converge.

The same ratio applies to the asymptotic position of the image charge. Taking $n \rightarrow \infty$ in expression (13) yields

$$z_{\infty} = a \lim_{n \rightarrow \infty} \left\{ \frac{\sinh(n\xi)}{\sinh[(n+1)\xi]} \right\} = a \lim_{n \rightarrow \infty} \left[\frac{e^{n\xi}}{e^{(n+1)\xi}} \right] = a e^{-\xi} . \quad (27)$$

The charges accumulate at this point inside $r < a$.

Limit for large heights

In the case $h \gg a$, and therefore $\xi \gg 1$, the series will converge very rapidly indeed. Summing the first three image charges, in eqs (1), (4), and (11), yields

$$C = 4\pi\epsilon_0 a \left[1 + \frac{a}{2h} + \frac{(a/2h)^2}{1 - (a/2h)^2} + \dots \right] . \quad (28)$$

For the extreme case, $h \rightarrow \infty$, we obtain the capacitance of an isolated sphere $C = 4\pi\epsilon_0 a$. As the sphere is brought near the ground its capacitance evidently increases. This happens because the charge on the sphere induces opposing charge in the ground, which in turn attracts more charge, of the original sign, in the sphere.

It turns out that in the limit $\xi \gg 1$ (i.e. $h \gg a$) the entire infinite series in eq. (24) can be summed exactly. In that limit we can approximate $\sinh(n\xi) \simeq e^{n\xi}/2$. Introducing this into the series gives

$$F(\xi) = \sum_{n=0}^{\infty} \frac{\sinh(\xi)}{\sinh[(n+1)\xi]} \simeq \sum_{n=0}^{\infty} e^{-n\xi} = \frac{1}{1 - e^{-\xi}} \quad , \quad (29)$$

where the final expression is the sum of a geometric series. Using the explicit form of \cosh^{-1} in eq. (12) gives

$$e^{\xi} = \frac{h}{a} + \sqrt{\frac{h^2}{a^2} - 1} \quad , \quad (30)$$

and

$$e^{-\xi} = \frac{h}{a} - \sqrt{\frac{h^2}{a^2} - 1} = \frac{h}{a} \left[1 - \sqrt{1 - \frac{a^2}{h^2}} \right] \quad . \quad (31)$$

Using this in (29) gives a version of the capacitance

$$C = 4\pi\epsilon_0 a \frac{a/h}{\sqrt{1 - (a/h)^2} - (1 - a/h)} \quad , \quad h \gg a \quad . \quad (32)$$

Some algebra confirms that the first terms of this expression match those from eq. (28). The expression diverges as $h \rightarrow a$, but this is outside its range of validity.

Limit for small heights

We have shown that the series in eq. (24) invariably converges, but for $\xi \ll 1$, it does not converge very quickly. Using the approximation $h/a = \cosh(\xi) \simeq 1 + \xi^2/2$ in eq. (12) gives

$$\xi \simeq \sqrt{\frac{2(h-a)}{a}} \ll 1 \quad , \quad (33)$$

so small values of ξ correspond to a small gap between conductor and ground: $h - a \ll a$. In this case the series

$$F(\xi) = 1 + \frac{\sinh(\xi)}{\sinh(2\xi)} + \frac{\sinh(\xi)}{\sinh(3\xi)} + \dots \simeq 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad , \quad (34)$$

will converge very slowly. If it continued in this manner it would ultimately diverge, however, at terms where $n\xi \gg 1$ relation (26) confirms that it will ultimately converge. Thus it would be necessary to keep at least $n \simeq 3/\xi$ terms or so before reaching a point of reasonable convergence. For small values of ξ this will be a very large number of terms.

As an alternative we write eq. (24) in the form

$$F(\xi) = 1 + \sinh(\xi) \sum_{s=2}^{\infty} \frac{1}{\sinh(s\xi)} , \quad (35)$$

where we have removed the first ($n = 0$) term and replaced the index with $s = n + 1$. The series cannot be summed exactly, but can be approximated by an integral which can be done

$$\sum_{s=2}^{\infty} \frac{1}{\sinh(s\xi)} \simeq \int_2^{\infty} \frac{ds}{\sinh(s\xi)} = \frac{1}{\xi} \ln \left[\tanh \left(\frac{1}{2} \xi s \right) \right] \Big|_{s=2}^{\infty} = -\frac{1}{\xi} \ln \left[\tanh(\xi) \right] . \quad (36)$$

For small values $\sinh(\xi) \simeq \tanh(\xi) \simeq \xi$ and the function

$$F(\xi) \simeq 1 - \ln(\xi) = -\ln(\xi/e) . \quad (37)$$

Using eq. (33) gives an approximate capacitance

$$C \simeq 2\pi\epsilon_0 a \ln \left[\frac{e^2 a}{2(h-a)} \right] , \quad (h-a) \ll a . \quad (38)$$

This diverges as the gap, $d = h - a$, vanishes. It is notable that it diverges logarithmically, $C \sim -\ln(d)$, while for a classic parallel plate capacitor the divergence is stronger, $C \sim 1/d$.

The discrepancy arises because only a limited region around the top of the sphere is really close to the ground plane. The radius r_{lim} at which the distance between sphere and ground is within $2d$ is set by the relation

$$\frac{r_{\text{lim}}^2}{a} \sim d .$$

The area of conductor inside this radius is $A \sim r_{\text{lim}}^2 \sim da$. As d decreases so does the area of this region — the region which resembles a parallel plate capacitor. The capacitance thus goes as $C \sim A/d \sim a d^0$, consistent with a $\ln(d)$ dependence, rather than the $1/d$ dependence it would have if the area A was independent of d .

How the trick was done

The key to finding the closed-form result was obtaining expression (13) for z_n . We demonstrated that this result satisfies the recursion relation, (7), and the initial condition, $z_0 = 0$. It is therefore the correct solution, and it is not necessary to know where it came from.

For those who are curious how one might go about solving such a non-linear recursion relation I give a few more details here. To simplify the calculation I define the dimensionless ratio

$$r_n = \frac{z_n}{a} , \quad (39)$$

in terms of which recursion relation (7) takes the form

$$r_{n+1} = \frac{1}{2(h/a) - r_n} . \quad (40)$$

This is *not* a linear relationship, and will therefore be difficult to work with. However, it takes the form of a ratio whose numerator and denominator are both linear in r_n (actually, the numerator is trivial, but this general procedure would work if it were even a more complicated linear function of r_n). To take advantage of this we write the unknown value r_n as a ratio of two unknown quantities

$$r_n = \frac{s_n}{t_n} , \quad (41)$$

Introducing this definition into (40) gives

$$\frac{s_{n+1}}{t_{n+1}} = \frac{1}{2(h/a) - s_n/t_n} = \frac{t_n}{2(h/a)t_n - s_n} . \quad (42)$$

There is some ambiguity in the definitions of s_n and t_n , which we can use to identify the numerator of the final expression with s_{n+1} , and the denominator with t_{n+1} . Each of these expressions is linear in the s_n and t_n . We can therefore write the recursion relation in linear form

$$\begin{bmatrix} s_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2(h/a) \end{bmatrix} \cdot \begin{bmatrix} s_n \\ t_n \end{bmatrix} = \underline{\underline{M}} \cdot \begin{bmatrix} s_n \\ t_n \end{bmatrix} . \quad (43)$$

We begin the iteration with $s_0 = 0$ and t_0 set to anything we wish — this is part of the arbitrariness. This linear relation is iterated by repeated multiplication by $\underline{\underline{M}}$

$$\begin{bmatrix} s_n \\ t_n \end{bmatrix} = \underline{\underline{M}}^n \cdot \begin{bmatrix} 0 \\ t_0 \end{bmatrix} . \quad (44)$$

The power of a matrix is easily found in terms of its eigenvalues and eigenvectors. The eigenvalues of $\underline{\underline{M}}$ are

$$\lambda_{1,2} = (h/a) \pm \sqrt{(h/a)^2 - 1} . \quad (45)$$

This expression is reminiscent of the analytic expression for \cosh^{-1} , and it may considerably be simplified by defining

$$\frac{h}{a} = \cosh(\xi) . \quad (46)$$

Using this in eq. (45) gives

$$\lambda_{1,2} = \cosh(\xi) \pm \sqrt{\cosh^2(\xi) - 1} = \cosh(\xi) \pm \sinh \xi = e^{\pm \xi} . \quad (47)$$

This simplification justifies the introduction of ξ in eq. (12). The corresponding eigenvectors are

$$\mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ e^{\pm \xi} \end{bmatrix} . \quad (48)$$

We may use these to define an initial vector

$$\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} = \mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \sinh(\xi) \end{bmatrix} . \quad (49)$$

having the required property, that $s_0 = 0$. The iterated form is

$$\begin{bmatrix} s_n \\ t_n \end{bmatrix} = \underline{\underline{M}}^n \cdot (\mathbf{v}_1 - \mathbf{v}_2) = (e^\xi)^n \mathbf{v}_1 - (e^{-\xi})^n \mathbf{v}_2 = \begin{bmatrix} e^{n\xi} - e^{-n\xi} \\ e^{(n+1)\xi} - e^{-(n+1)\xi} \end{bmatrix} . \quad (50)$$

To obtain the result we simply take the ratio of the vector components

$$\frac{z_n}{a} = r_n = \frac{s_n}{t_n} = \frac{e^{n\xi} - e^{-n\xi}}{e^{(n+1)\xi} - e^{-(n+1)\xi}} = \frac{\sinh(n\xi)}{\sinh[(n+1)\xi]} , \quad (51)$$

matching the result presented in eq. (13).