

Notes on Transmission Lines

(DWL 4/10/01, significantly revised 4/2/19, and then 4/16/20)

Wave equations occur in many areas of physics but electromagnetic waves are probably the most widely known. The three-dimensional wave equation which describes electromagnetic radiation in vacuum is quite formidable. It is worthwhile becoming acquainted with wave equations in one dimension before tackling the three-dimensional variety.

Electrical signals propagating along a wire obey a one-dimensional wave equation. These notes derive the equations from the underlying physics, also called *transmission line theory*, and present the basic properties of a wave equation in this context. As an additional benefit, several results from transmission line theory may prove useful to anyone needing to connect computer cables or work with electronics in the lab.

1.1 The transmission line

A particularly simple kind of transmission line is the co-axial cable composed of two conductors. We will consider a perfectly straight co-axial conductor whose inner conductor is a cylinder of outer radius a centered on the z -axis. This is surrounded by an outer shielding conductor whose inner radius is b and also centered on the z axis — i.e. co-axial. Both conductors will be assumed to be perfectly conducting ($\sigma = \infty$); this theoretical construct is known as a *lossless transmission line*. (A section below relaxes this treatment to include conductors with finite resistivity — a *lossy transmission line*.) The entire gap is filled with a material with linear electric and magnetic properties, described by permittivity ϵ and permeability μ . A general signal on the co-axial cable consists of a current $I(z, t)$ (Amps) and a line-charge $\lambda(z, t)$ (Coulombs per meter) on the inner conductor. We will assume that the outer conductor carries an equal and opposite current and charge. Because we are assuming perfect conductors the current density and charge density will be distributed only over the surfaces of each conductor. $I(z, t)$ and $\lambda(z, t)$ are the integrals over these surface densities.

Electrostatic field

We compute the electric field within the gap as the sum of an electrostatic and an inductive contribution

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_s(\mathbf{x}, t) + \mathbf{E}_i(\mathbf{x}, t) . \quad (1)$$

The electrostatic field arises only from free and bound surface charges on the inner conductor and inner surface of the dielectric. Provided $\lambda(z, t)$ varies over axial length $\Delta z \gg b$, the field will be *primarily* radial and we may write the electric displacement

$$\mathbf{D}(s, z, t) \simeq \frac{\lambda(z, t)}{2\pi\rho} \hat{\rho} . \quad (2)$$

The electrostatic field is therefore

$$\mathbf{E}_s(s, z, t) = \frac{\mathbf{D}}{\epsilon} \simeq \frac{\lambda(z, t)}{2\pi\epsilon\rho} \hat{\rho} . \quad (3)$$

Since this is an electrostatic field, it can be written as the gradient of a scalar potential

$$\Phi(\rho, z, t) \equiv - \int \mathbf{E}_s \cdot d\mathbf{l} = - \int_b^\rho E_{s,\rho}(\rho', z, t) d\rho' = \frac{\ln(b/\rho)}{2\pi\epsilon} \lambda(z, t) , \quad (4)$$

where the potential of the outer conductor has been taken to be zero. Because Φ depends on z there is also an axial component to the electrostatic field

$$E_{s,z} = -\frac{\partial\Phi}{\partial z} = -\frac{\ln(b/\rho)}{2\pi\epsilon} \frac{\partial\lambda}{\partial z} , \quad (5)$$

which appears to be missing from the expression (3). The reason for this discrepancy is that the ratio of the two is very small

$$\left| \frac{E_{s,z}}{E_{s,\rho}} \right| \simeq \left| \frac{\partial\lambda}{\partial z} \ln(b/\rho) \frac{\rho}{\lambda} \right| \sim \ln(b/\rho) \frac{\rho}{\Delta z} \ll 1 , \quad (6)$$

where we define the length scale of axial variation as

$$\Delta z \sim \left| \frac{1}{\lambda} \frac{\partial\lambda}{\partial z} \right|^{-1} . \quad (7)$$

We started off assuming this length scale to be much larger than b , and therefore much larger than all values of ρ .¹ Thus the purely radial electrostatic field in eq. (3) is the leading order approximation, and we used it to define the potential, in eq. (4). It is from that expression that we derive the smaller axial component in eq. (5).

The voltage at the inner conductor is found by evaluating the potential at $\rho = a$ there

$$V(z, t) = \Phi(a, z, t) = \frac{\ln(b/a)}{2\pi\epsilon} \lambda(z, t) . \quad (8)$$

Since we have taken $\Phi = 0$ at the outer conductor, this is the voltage drop between the inner and outer conductor. The voltage is evidently proportional to the charge per length on that conductor so

$$\lambda(z, t) = \underbrace{\frac{2\pi\epsilon}{\ln(b/a)}}_c V(z, t) , \quad (9)$$

where the constant of proportionality is the *capacitance per unit length*² of the co-axial cable

$$c = \frac{2\pi\epsilon}{\ln(b/a)} . \quad (10)$$

This quantity will play a central role in our computations below. Had we used conductors of any geometry other than co-axial cylinders, we would be able (possibly with difficulty) to compute the capacitance between them. Provided they were invariant in z , the total capacitance would be proportional to total length, with a constant of proportionality \mathcal{C} , the capacitance per unit length.

¹This is equivalent to restricting ourselves to frequencies much smaller than the inverse light-transit time across the cable's diameter. For cable diameters of order 1 cm this means restricting ourselves to frequencies below a few GHz. At higher frequencies the system must be treated as a *wave-guide* rather than a transmission line.

²Obviously if the ratio of charge to voltage is capacitance then the ratio of charge per length to voltage is capacitance per length. Its units are, naturally, Farads per meters, which happens to be same as the units of permittivity: ϵ_0 and ϵ .

Inductive electric field

Using similar arguments about the scale of axial variations of the current $\Delta z \gg b$, the magnetic field can be seen to be primarily in the $\hat{\phi}$ direction, and may be found from Ampère's law

$$\mathbf{B}(\rho, z, t) \simeq \frac{\mu I(z, t)}{2\pi\rho} \hat{\phi} , \quad (11)$$

where we have used the fact that $\mathbf{B} = \mu\mathbf{H}$ in the linear medium. This magnetic field can be expressed using a purely axial vector potential

$$\mathbf{A}(\rho, z, t) = A_z(\rho, z, t) \hat{\mathbf{z}} , \quad (12)$$

by taking the curl

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla A_z \times \hat{\mathbf{z}} = -\frac{\partial A_z}{\partial \rho} \hat{\phi} . \quad (13)$$

Equating this with eq. (11) allows us to obtain the vector potential explicitly

$$A_z(\rho, z, t) = -\int_b^\rho B_\phi(\rho', z, t) d\rho' = -\int_b^\rho \frac{\mu I(z, t) d\rho'}{2\pi\rho'} = \frac{\mu I(z, t)}{2\pi} \ln(b/\rho) , \quad (14)$$

where we have used our gauge freedom to set $A_z = 0$ on the outer conductor (but we see below that this is necessary to be consistent with our choice of $\Phi = 0$ there).

At this point we will take a brief tangent to rephrase the result in terms of an inductance per unit length of the co-axial cable. To do so we introduce an Ampèrian loop, AL , shown in fig. 1. The loop crosses from the inner to outer conductor but extends only a short axial distance $\delta z \ll \Delta z$ the length over which $I(z, t)$ varies. The flux through that small loop is

$$\delta F = \oint_{AL} \mathbf{A} \cdot d\mathbf{l} = \int_z^{z+\delta z} [A_z(a, z', t) - A_z(b, z', t)] dz' \simeq A_z(a, z, t) \delta z , \quad (15)$$

after noting that $A_z = 0$ at $\rho = b$, and using the fact that $A_z(a, z', t)$ is approximately constant over the short distance. The result is that the flux per unit axial length is

$$\frac{\delta F}{\delta z} = A_z(a, z, t) = \underbrace{\frac{\mu}{2\pi} \ln(b/\rho)}_{\mathcal{L}} I(z, t) , \quad (16)$$

where we recognize the constant of proportionality between current and flux-per-length, as inductance per length

$$\mathcal{L} = \frac{\mu}{2\pi} \ln(b/a) . \quad (17)$$

Comparing this expression for inductance per length to the capacitance per length in eq. (10) reveals the relation

$$\mathcal{L} = \frac{\mu\epsilon}{C} , \quad (18)$$

for a co-axial cable. This turns out to be true for two axially-invariant conductors *regardless* of their cross sectional shapes — not just for co-axial cylinders. It is not trivial to show, but follows from

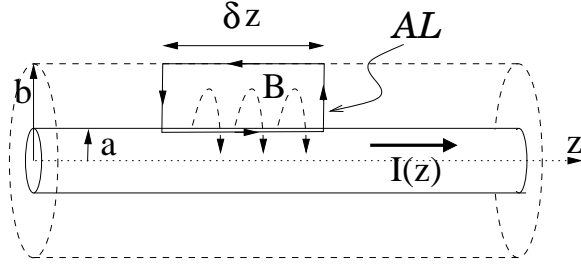


Figure 1: The Ampèrian loop, AL , used to compute inductance per unit length.

the fact that Φ and A_z both satisfy Laplace's equation between the conductors, and that λ and I arise from integrals of their respective normal derivatives over the conductor surface. Problem 5.29 in Jackson asks for these steps to be filled in explicitly — have fun.

The final step in this analysis is to use A_z to compute the inductive electric field between the conductors

$$E_{i,z}(\rho, z, t) = -\frac{\partial A_z}{\partial t} . \quad (19)$$

This must be added to the axial component of the electrostatic field, found in eq. (5), to obtain the full electric field

$$E_z(\rho, z, t) = -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial t} . \quad (20)$$

Evaluating this at $\rho = a$ and $\rho = b$ gives the *tangential electric field* outside each conductor. Since both are perfect conductors the field must vanish at those points. Since we have chosen both $\Phi = 0$ and $A_z = 0$ at the outer boundary, $\rho = b$, we automatically satisfy $E_z = 0$ there. To satisfy the condition on the inner conductor we demand

$$\frac{\partial \Phi(a, z, t)}{\partial z} = -\frac{\partial A_z(a, z, t)}{\partial t} . \quad (21)$$

Using eqs. (8) and (16) to replace $\Phi(a, z, t)$, and $A_z(a, z, t)$, gives

$$\frac{\partial V}{\partial z} = -\mathcal{L} \frac{\partial I}{\partial t} . \quad (22)$$

This gives one fundamental relation between the voltage and current. The relation reveals that even though the inner conductor is a perfect conductor, it is *not* an equipotential: i.e. $\partial V/\partial z \neq 0$. Conductors are equipotential in electrostatics because there is no inductive contribution to the electric field. When there is an inductive contribution then it must be canceled by a non-vanishing electrostatic field at the conductor.

To obtain a second relation between the voltage and current we now introduce the continuity of charge. A variation in current along the inner conduct, i.e. $\partial I/\partial z \neq 0$, results in a change in the charge density there. Conservation of charge demands that

$$\frac{\partial \lambda}{\partial t} = -\frac{\partial I}{\partial z} . \quad (23)$$

Using eq. (9) to write $\lambda = \mathcal{C}V$ we obtain the explicit relation

$$\mathcal{C} \frac{\partial V}{\partial t} = -\frac{\partial I}{\partial z} . \quad (24)$$

1.2 The wave equation

Equations (22) and (24) constitute two first-order partial differential equations (PDEs) relating the evolutions of current and voltage distributions on the inner conductor of our co-axial cable. The pair of equations, referred to collectively as *telegrapher's equations*, are traditionally written to give time-dependences explicitly

$$\frac{\partial I}{\partial t} = -\frac{1}{\mathcal{L}} \frac{\partial V}{\partial z} , \quad (25)$$

$$\frac{\partial V}{\partial t} = -\frac{1}{\mathcal{C}} \frac{\partial I}{\partial z} . \quad (26)$$

The telegrapher's equations may be combined into a single second-order PDE for a single variable. One way to do this is differentiating (25) by z , (26) by t and eliminating the cross derivative $\partial^2 I / \partial z \partial t$. The resulting equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{\mathcal{L}\mathcal{C}} \frac{\partial^2 V}{\partial z^2} = c^2 \frac{\partial^2 V}{\partial z^2} , \quad (27)$$

is the one-dimensional wave equation with wave-speed

$$c = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\epsilon\mu}} , \quad (28)$$

after making use of the general relation given by eq. (18). This is the speed of light in the linear medium. Evidently, voltage signals propagate along a co-axial cable, or any other transmission line, at the speed of light.

The general solution to (27) can be written as the sum of rightward and leftward propagating wave-forms

$$V(z, t) = f(t - z/c) + g(t + z/c) , \quad (29)$$

where $f(u)$ and $g(v)$ are arbitrary functions of a single variable (see fig. 2). To verify that expression (29) solves the wave equation, (27), we must work out the partial derivatives using the chain rule

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{df}{du} \frac{\partial u}{\partial t} + \frac{dg}{dv} \frac{\partial v}{\partial t} \\ &= f'(t - z/c) + g'(t + z/c) \end{aligned}$$

$$\begin{aligned} \frac{\partial V}{\partial z} &= \frac{df}{du} \frac{\partial u}{\partial z} + \frac{dg}{dv} \frac{\partial v}{\partial z} \\ &= -c^{-1} f'(t - z/c) + c^{-1} g'(t + z/c) \end{aligned}$$

where $f'(u) \equiv df/du$ etc. Repeating this step gives the second derivatives

$$\begin{aligned} \frac{\partial^2 V}{\partial t^2} &= f''(t - z/c) + g''(t + z/c) \\ \frac{\partial^2 V}{\partial z^2} &= c^{-2} f''(t - z/c) + c^{-2} g''(t + z/c) , \end{aligned}$$

and equation (27) can be seen to be satisfied regardless of how functions $f(u)$ and $g(v)$ are chosen.

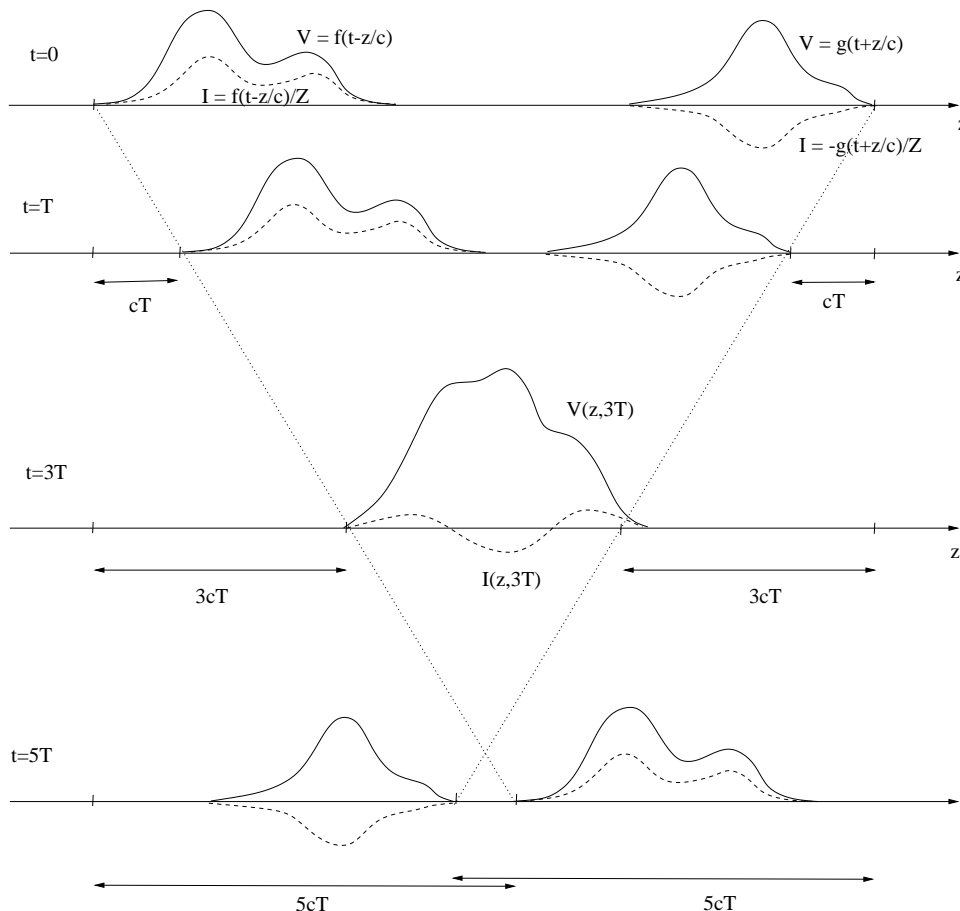


Figure 2: Rightward and leftward propagating pulses on a transmission line. The potential $V(z, t)$ and current $I(z, t)$ are shown at times $t = 0, T, 3T$ and $5T$.

Consider a case where one end of the cable, at $z = 0$, is connected to a transmitter which generates a voltage signal

$$V(z=0, t) = S(t) . \tag{30}$$

This can be applied as a boundary condition on eq. (29) to yield a full solution

$$V(z, t) = S(t - z/c) , \tag{31}$$

which is purely rightward propagating. A receiver located at position z will observe a voltage on the line which is the same as that at the generator but delayed³ by time z/c ; this is the signal transit time.

The current can be found by substituting the general solution (29) into eq. (25)

$$\frac{\partial I}{\partial t} = -\frac{1}{\mathcal{L}} \frac{\partial V}{\partial z} = -\frac{1}{c\mathcal{L}} [-f'(t - z/c) + g'(z + ct)] \tag{32}$$

³The voltage applied at $t = 0$, namely $S(0)$ is observed by the receiver at $t = z/c$ since that is when the argument of (31) is zero.

$$= \frac{1}{c\mathcal{L}} \frac{\partial}{\partial t} [f(t - z/c) - g(t + z/c)] . \quad (33)$$

In the last step, the derivatives of f and g have been changed to partial derivatives with respect to time. This permits the equation to be easily solved for current

$$I(z, t) = \frac{1}{c\mathcal{L}} [f(t - z/c) - g(t + z/c)] = Z^{-1} [f(t - z/c) - g(t + z/c)] , \quad (34)$$

where the *impedance* of the transmission line

$$Z = c\mathcal{L} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} , \quad (35)$$

has been introduced to simplify notation. Note that the current also takes the form of rightward and leftward traveling waveforms. Moreover, the shapes of the current waveforms exactly match those of the voltage: namely $f(u)$ and $g(v)$. In a rightward waveform the voltage and currents have the same signs while in a leftward waveform they have opposite signs.

For a purely rightward (top) or leftward (bottom) propagating wave the voltage and current are related by

$$V = \pm Z I , \quad (36)$$

where the impedance Z apparently has units of resistance. This simple relation *does not* hold in general; a transmission line is not a resistor.

Impedance

We claimed above, with a brief nod toward a justification, that the product $\mathcal{L}\mathcal{C} = \epsilon\mu$ is independent of the geometry of the conductors. The ratio of the two *will*, however, depend on geometry, and so will the impedance. Using expressions (10) and (17) yields the impedance of a co-axial cable

$$Z = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu}{\epsilon}} . \quad (37)$$

The dimensional piece of the impedance is $\sqrt{\mu/\epsilon}/2\pi$ has units of resistance. Its value in vacuum $\sqrt{\mu_0/\epsilon_0}/2\pi = 60$ Ohms, is determined by universal constants, and sometimes called *the impedance of vacuum*.⁴ Typically $\mu \simeq \mu_0$ and $\epsilon > \epsilon_0$ so the wave speed is less than c , and the impedance is smaller than its value would be with a vacuum.

The factor $\ln(b/a)$ arises in eq. (37) from the co-axial geometry. It could have been quickly found by computing either \mathcal{C} or \mathcal{L} , for co-axial conductors. For a transmission line with any other shape one must repeat that step to find its impedance. One other common example is a pair of identical wires, each of radius a and separated by center-to-center distance d . This is a variant of Jackson's problem 5.26, with current on the outside rather than uniformly distributed. The inductance per unit length is

$$\mathcal{L}_{\text{wires}} = \frac{\mu}{\pi} \ln(d/a) . \quad (38)$$

⁴That vacuum has finite impedance, while it clearly has infinite resistance, reminds us once more than impedance and resistance are different and independent things — they just happen to have the same units.

Multiplying this by $c = 1/\sqrt{\epsilon\mu}$ gives the impedance of the parallel wires

$$Z_{\text{wires}} = c\mathcal{L}_{\text{wires}} = \frac{\ln(d/a)}{\pi} \sqrt{\frac{\mu}{\epsilon}} . \quad (39)$$

This resembles the impedance of a co-axial cable, i.e. eq. (37), with $d = b$, but is a factor of two larger. This is basically the penalty for filling all space with electric and magnetic fields, rather than just an annular gap.

50 Ohm impedance is a common standard for co-axial cables. If such a co-axial cable had a vacuum gap the outer radius would be $b = 2.3a$. One sample of 50Ω co-axial cable actually has a wave speed $0.80c$, which means that the permittivity in the gap is $\epsilon = (0.80)^{-2}\epsilon_0$ ($\mu \simeq \mu_0$ in non-ferromagnetic materials). From this we can infer the outer gap is $b = 2.8a$.

1.3 Boundary conditions

Resistive terminations

Consider a transmission line of length ℓ at the end of which, i.e. at $z = \ell$, the inner and outer conductors are connected together by a resistor R . This imposes the condition

$$V(\ell, t) = RI(\ell, t) , \quad (40)$$

for all time t . Substituting both general solutions, (29) and (34), we find

$$f(t - \ell/c) + g(t + \ell/c) = \frac{R}{Z}[f(t - \ell/c) - g(t + \ell/c)] . \quad (41)$$

Solving this for the reflected wave g gives the relation

$$g(t + \ell/c) = \frac{R - Z}{R + Z} f(t - \ell/c) . \quad (42)$$

It is possible to eliminate all reflections if the terminating resistor is chosen to exactly match the impedance Z of the transmission line. A larger resistance will result in a reflection without sign flip, a smaller resistance will result in a reflection with a sign flip. The cases of open and shorted ends can be easily recovered by setting $R = \infty$ and $R = 0$ respectively in expression (42).

It is useful to find a general expression for the function $g(v)$. To do this we equate the argument of eq. (42) with the variable $v = t + \ell/c$. If we invert this relation we find $t = v - \ell/c$. Using that to eliminate t from the argument of f yields

$$g(v) = \frac{R - Z}{R + Z} f(v - 2\ell/c) . \quad (43)$$

This is what is used in the general solution, eq. (29) to get

$$V(z, t) = f(t - z/c) + \underbrace{g(t + z/c)}_v = f(t - z/c) + \frac{R - Z}{R + Z} f\left(t - \frac{2\ell - z}{c}\right) . \quad (44)$$

This shows that position z sees a direct signal delayed by z/c , and then a reflected signal delayed by $(2\ell - z)/c$. Since $z < \ell$ within the line, the reflection will always arrive *after* the direct signal — naturally. The reflection will reach the origin, $z = 0$, delayed by a full round-trip: $2\ell/c$.

1.4 Connecting two cables

Consider the case of two cables with different impedances connected together. One co-axial cable with impedance Z_1 and wave speed c_1 stretches from $z = 0$ to $z = \ell$ where it is connected to a cable with impedance Z_2 and wave speed c_2 . A rightward signal $f_1(t - z/c_1)$ reaching this juncture will give rise to a rightward signal in cable 2, which we denote $f_2(t - z/c_2)$, and a reflected leftward signal on cable 1, $g_1(t + z/c_1)$. The “connection” consists of electrical contact between both the inner and outer conductors of the cables. The voltage across inner and outer conductors must match across the juncture

$$V(\ell - 0, t) = f_1(t - \ell/c_1) + g_1(t + \ell/c_1) = f_2(t - \ell/c_2) = V(\ell + 0, t) . \quad (45)$$

The currents must also match, since no charge can disappear into the connection

$$Z_1^{-1} [f_1(t - \ell/c_1) - g_1(t + \ell/c_1)] = Z_2^{-1} f_2(t - \ell/c_2) . \quad (46)$$

Eliminating f_2 from this pair of equations we find the reflected wave

$$g_1(t + \ell/c_1) = \frac{Z_2 - Z_1}{Z_2 + Z_1} f_1(t - \ell/c_1) , \quad (47)$$

appears exactly as if it had reflected from a lumped resistance $R = Z_2$. Replacing this in (45) yields the transmitted wave

$$f_2(t - \ell/c_2) = \frac{2Z_2}{Z_2 + Z_1} f_1(t - \ell/c_1) . \quad (48)$$

The waveform is transmitted to cable 2 multiplied by a factor ranging from zero to two; it never flips sign. It is notable that any change in wave speed between the cables plays no role in reflection or transmission.

1.5 An illustrative example

As an example consider a cable with impedance Z terminated with a resistor R which, for concreteness, we take to be $R < Z$. At $t = 0$ the cable’s left end is connected to a battery with voltage V_0 . This causes a step-function pulse to travel rightward, as illustrated in fig. 3. The current is also a step function with current

$$I_0 = \frac{V_0}{Z} . \quad (49)$$

This is the current flowing across the switch for some time after it is closed. Note that it is unrelated to the value of the resistor. This is natural, since the resistor is some distance away and the battery cannot yet know about it.

When the pulse reaches the resistor, at $t = c/\ell$, it produces a reflected pulse. The voltage in this pulse is

$$\Delta V_1 = \frac{R - Z}{Z + R} V_0 , \quad (50)$$

as required by eq. (42). Since $R < Z$, the reflected voltage is negative, $\Delta V_1 < 0$, and thus reduces the total voltage (see fig. 3). Since it is a leftward wave it contributes a positive current

$$\Delta I_1 = -\frac{\Delta V_1}{Z} = \frac{Z - R}{Z + R} \frac{V_0}{Z} = \alpha I_0 . \quad (51)$$

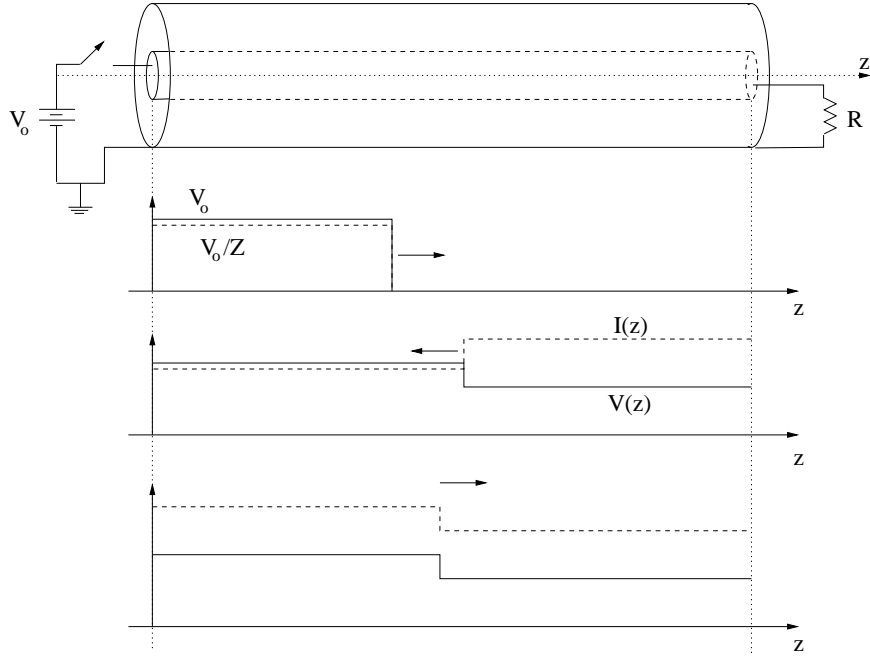


Figure 3: A battery is connected to a transmission line with resistor $R < Z$. Initially a voltage and current pulse is transmitted rightward. A reflection from the resistor propagates leftward. This is then reflected rightward by the battery.

where $\alpha = (Z - R)/(Z + R)$. This increases the current in order to satisfy the condition

$$V_0 + \Delta V_1 = R(I_0 + \Delta I_1) \quad , \quad (52)$$

at $z = \ell$.

When the leftward reflection reaches the origin, at time $t_1 = 2\ell/c$, a new rightward reflection is then generated. This is needed because the battery must maintain $V = V_0$ at $z = 0$ — this is the one and only job of a battery. The reflected voltage pulse will be equal and opposite to the incident voltage in order to preserve $V = V_0$. The reflected current is therefore positive — again. The current through the switch will then be

$$I_1 = I_0 + 2\Delta I_1 = I_0 + 2\alpha I_0 \quad , \quad t_1 < t < t_2 = 2t_1 \quad . \quad (53)$$

This is the result after one round-trip reflection, and it will persist until a new reflection arrives at $t_2 = 4\ell/c$.

This new rightward reflection will suffer another reflection at the resistor generating

$$\Delta I_2 = \alpha \Delta I_1 = \alpha^2 I_0 \quad ; \quad (54)$$

all reflections have positive current and thus cause the net current to monotonically increase. The voltage reflections, on the other hand, are positive going rightward and negative going leftward. They cause the voltage to oscillate. The reflections from the left are always such that they restore $V = V_0$ at $z = 0$.

It is evident how this procedure might be repeated to obtain the current at the switch after n round-trip reflections

$$I_n = \left[1 + 2\alpha + 2\alpha^2 + \cdots + 2\alpha^n \right] I_0 \quad , \quad t_n < t < t_{n+1} \quad , \quad (55)$$

illustrated in fig. 4. Since $\alpha < 1$ (except in the troubling case $R = 0$) the series converges to an asymptotic current

$$I_\infty = \left[1 + 2 \sum_{n=1}^{\infty} \alpha^n \right] I_0 = \left[1 + \frac{2\alpha}{1-\alpha} \right] I_0 = \frac{1+\alpha}{1-\alpha} I_0 \quad , \quad (56)$$

after summing the geometric series. Introducing $\alpha = (Z - R)/(Z + R)$ yields simple form for the factor

$$\frac{1+\alpha}{1-\alpha} = \frac{(Z+R) + (Z-R)}{(Z+R) - (Z-R)} = \frac{Z}{R} \quad . \quad (57)$$

The current therefore asymptotes to

$$I_\infty = \frac{1+\alpha}{1-\alpha} I_0 = \frac{Z}{R} I_0 = \frac{Z}{R} \left(\frac{V_0}{Z} \right) = \frac{V_0}{R} \quad . \quad (58)$$

the value the resistor would draw if it were connected to the battery by two perfectly conducting wires. Of course the transmission line is just that: two perfectly conducting wires.

The novel aspect seen in this example is that the current V_0/R , is not achieved immediately upon closing the switch. Instead it is approached asymptotically through repeated reflection between battery and resistor as seen in fig. 4. This process seems like a stair-step rendering of the asymptotic current ramp-up observed in a classic L-R circuit, shown as a dashed curve. In that case, the current approaches its asymptotic value exponentially. The geometric progression we have found is, in fact, similar to an exponential with time constant τ given the association

$$\alpha^n \sim e^{-t_n/\tau} \quad , \quad (59)$$

where $t_n = 2n\ell/c$ is the time taken for n round-trip reflections. Taking logarithms we find an expression for the time constant

$$\tau = - \frac{t_n}{n \ln \alpha} = \frac{2\ell}{c \ln(\alpha^{-1})} \quad . \quad (60)$$

In the cases where R and Z have comparable magnitudes (but $R < Z$) then $\ln(\alpha^{-1})$ will not be very small and $\tau \sim 2\ell/c$ — it will take only a few transits for the current to approach its asymptote.

In the opposite case of extremely small resistance, $R \ll Z$, each reflection will be very small, $I_\infty \gg I_0$, so it will take many to approach the asymptotic current. In this limit we expand

$$\alpha^{-1} = \frac{Z+R}{Z-R} = \left(1 + \frac{R}{Z} \right) \left(1 - \frac{R}{Z} \right)^{-1} \simeq 1 + \frac{2R}{Z} + \cdots \quad , \quad (61)$$

and the very small logarithm

$$\ln(\alpha^{-1}) \simeq \ln \left(1 + \frac{2R}{Z} + \cdots \right) \simeq \frac{2R}{Z} \quad . \quad (62)$$

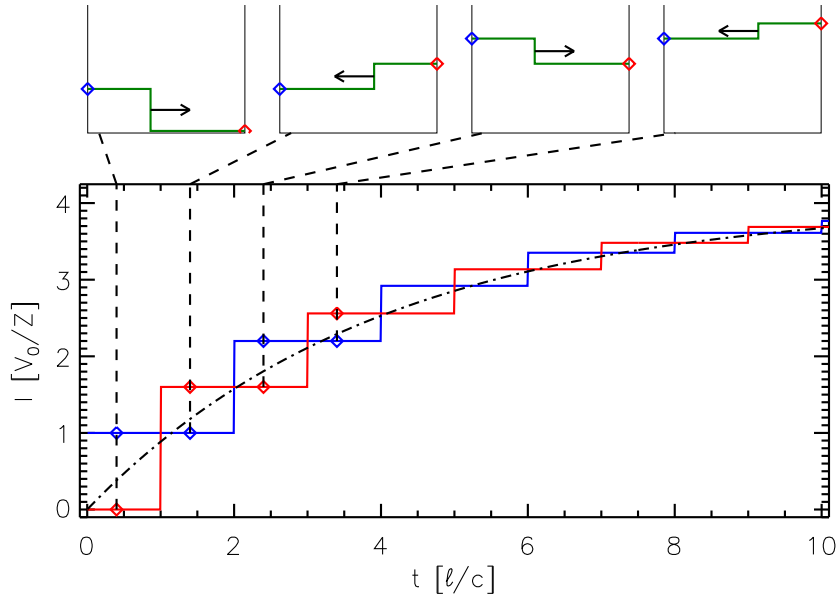


Figure 4: The current in a transmission line with a resistive load $R = Z/4$ at $z = \ell$ and a voltage V_0 applied to $z = 0$ at $t = 0$. The bottom plot shows current at $z = 0$ (blue) and $z = \ell$ (red) *vs.* time, in units of transit-time ℓ/c . The black dashed curve is the current in an L-R circuit, asymptoting to $I_\infty = 4V_0/Z$. The upper plots show the current $I(z)$ at times during the first four half-transits, with red and blue symbols at the appropriate ends, and a black arrow indicating the direction the transition is propagating.

In this case the time constant becomes much larger

$$\tau = \frac{2\ell}{c \ln(\alpha^{-1})} \simeq \frac{\ell Z}{cR} = \frac{\ell \mathcal{L}}{R} , \quad (63)$$

after using $Z = c\mathcal{L}$. Since \mathcal{L} is the inductance per unit length, the product $\ell\mathcal{L} = L$ is the *total* inductance of the entire transmission line. The time constant, $\tau = L/R$, is exactly what is found for a simple L-R circuit. It would seem that when $R \ll Z$ the transmission line behaves like an inductor. The current rises gradually because of the inductance of the wire, but if observed closely it does so in a stair-step manner.

We can use the result, eq. (63), to re-express the assumption as

$$\frac{R}{Z} \simeq \frac{\ell/c}{\tau_{LR}} \ll 1 . \quad (64)$$

The middle ratio says that the light-transit time across the transmission line is much smaller than the L-R time-constant. If this is the case, the transmission line will behave like a simple inductor; we call this a *lumped circuit element*, and it is what you studied in undergraduate electrical circuits. The treatment breaks down when the lumped-circuit time constant, i.e. $\tau = L/R$, becomes comparable to light-transit across the circuit. At that point it becomes necessary to account for currents and voltages traveling from one part of the circuit to another — i.e. to use transmission

line theory. Since you were typically thinking of circuits only a foot or less across, a lumped-circuit approach is valid provided it is not operating at giga-Hertz frequencies.

While we had described the process for a resistance, $R < Z$, the same process can be applied to the case $R > Z$. Indeed, the results above hold for both cases, although when $R > Z$ the factor $\alpha = (Z - R)/(Z + R)$ will be negative. Using this in eq. (53) gives $I_1 < I_0$. Following this logic leads to $I_2 > I_1$, $I_3 < I_2$, etc. The asymptotic current $I_\infty < I_0$ is thus approached via oscillation, rather than monotonically as in the original version (i.e. $R < Z$). This is a natural consequence of the successive powers of a negative factor: α^n . This power can still be expressed as $\alpha^n = e^{-t_n/\tau}$, although the time constant is now complex. Using the reciprocal of expression (60) results in the complex value

$$\frac{1}{\tau} = \frac{c}{2\ell} \ln(\alpha^{-1}) = \frac{c}{2\ell} \ln\left(\frac{R+Z}{R-Z}\right) \pm i\frac{\pi c}{2\ell} , \quad (65)$$

where the imaginary part is from $\ln(-1) = \pm i\pi$. The imaginary part of this leads to a complex phase, $e^{-t/\tau} \propto e^{-\omega t}$ with angular frequency

$$\omega = \frac{\pi c}{2\ell} , \quad (66)$$

meaning a period of $4\ell/c$. The reason for this is that the voltage pulse is inverted by the left boundary (the voltage source at $z = 0$) but not by the resistive load on the right when $R > Z$. It therefore takes two full transits before the pulse returns to its original sign, leading to a period $4\ell/c$.

Since the real part of expression (65) is positive, current still approaches I_∞ as $t \rightarrow \infty$. In the case of very great load resistance, $R \gg Z$, we can expand this to obtain

$$\frac{1}{\tau_r} = \frac{c}{2\ell} \ln\left(\frac{R+Z}{R-Z}\right) \simeq \frac{cZ}{\ell R} = \frac{1}{R\ell C} , \quad (67)$$

which is the inverse of the RC time constant (the total capacitance of the transmission line is $C = \ell C$). This decaying oscillation suggests that the loaded transmission line behaves like an LRC circuit with the resistor and capacitor in parallel. The long time behavior of this simple circuit is dominated by exponential decay at the longer of its time constants. This is $\tau \simeq L/R$ when, $R \ll \sqrt{L/C}$, and is $\tau \simeq RC$ when $R \gg \sqrt{L/C}$. In the former case the decay is slow enough ($\tau = L/R \gg RC$) that the capacitor behaves like an open circuit leaving an LR circuit with its simple exponential decay. In the latter case, however, the decaying envelope oscillates at the intermediate frequency $1/\sqrt{LC}$. Both behaviors are found in the loaded transmission line, although that exhibits oscillation at a frequency smaller by a factor $2/\pi$ than the circuit.

1.6 Circuit impedance

Classic lumped circuit analysis invokes impedance in a different, but related manner. If a particular lumped circuit element carries current $I(t) = I_0 e^{-i\omega t}$, the voltage drop across it will be $V(t) = \tilde{Z}(\omega) I_0 e^{-i\omega t}$, where $\tilde{Z}(\omega)$ is its impedance. Resistors, capacitors, and inductors have impedances

$$\tilde{Z}_R(\omega) = R \quad , \quad \tilde{Z}_C(\omega) = \frac{i}{\omega C} \quad , \quad \tilde{Z}_L(\omega) = -i\omega L \quad , \quad (68)$$

respectively. The circuit impedance for any combination of elements is found by combining their impedances just as one combines resistances — simple addition for series combinations and adding reciprocals to find the reciprocal impedance of a parallel combination.

To compute the circuit impedance of a transmission line we consider one driven by a current $I(0, t) = I_0 e^{-i\omega t}$ at one end — the $z = 0$ end. The general solution to the wave equation with that boundary condition is

$$I(z, t) = I_0 \cos(kz) e^{-i\omega t} + \tilde{A} \sin(kz) e^{-i\omega t} , \quad (69)$$

where $k = \omega/c$ and \tilde{A} is a constant. To find the voltage we use eq. (26)

$$\frac{\partial V}{\partial t} = -\frac{1}{\mathcal{C}} \frac{\partial I}{\partial z} = \omega Z [I_0 \sin(kz) - \tilde{A} \cos(kz)] e^{-i\omega t} , \quad (70)$$

after using the fact that $1/c\mathcal{C} = c\mathcal{L} = Z$, the transmission line impedance (the tilde is used to distinguish the circuit impedance from this one). This gives us a voltage

$$V(z, t) = iZ [I_0 \sin(kz) - \tilde{A} \cos(kz)] e^{-i\omega t} . \quad (71)$$

If the other end of this transmission line, $z = \ell$, is connected to a load impedance \tilde{Z}_ℓ then we must have

$$\begin{aligned} V(\ell, t) &= iZ [I_0 \sin(k\ell) - \tilde{A} \cos(k\ell)] e^{-i\omega t} \\ &= \tilde{Z}_\ell I(\ell, t) = \tilde{Z}_\ell [I_0 \cos(k\ell) + \tilde{A} \sin(k\ell)] e^{-i\omega t} , \end{aligned} \quad (72)$$

This relation can be solved to find

$$\tilde{A} = \left[\frac{iZ \tan(k\ell) - \tilde{Z}_\ell}{\tilde{Z}_\ell \tan(k\ell) + iZ} \right] I_0 . \quad (73)$$

To finish we use eq. (71) to obtain the voltage at $z = 0$

$$V(0, t) = -iZ \tilde{A} e^{-i\omega t} = \tilde{Z}(\omega) I_0 e^{-i\omega t} , \quad (74)$$

given the circuit impedance of the load through the transmission line

$$\tilde{Z}(\omega) = -iZ \left[\frac{iZ \tan(k\ell) - \tilde{Z}_\ell}{\tilde{Z}_\ell \tan(k\ell) + iZ} \right] = Z \left[\frac{Z \tan(\omega\ell/c) + i\tilde{Z}_\ell}{\tilde{Z}_\ell \tan(\omega\ell/c) + iZ} \right] . \quad (75)$$

This is more complicated than the simple circuit elements in eq. (68). It does have several properties of note. If the transmission line is negligibly short, $\omega\ell/c \rightarrow 0$, then it does not effect the combination and $\tilde{Z}(\omega) = \tilde{Z}_\ell$. If the load impedance exactly matches the transmission line, $\tilde{Z}_\ell = Z$, then the combination matches that value as well $\tilde{Z}(\omega) = Z = \tilde{Z}_\ell$. This is notable, because it appears that the transmission line plays no role, even though it may be very long.

Slightly more informative is the case, treated above, where $\tilde{Z}_\ell \ll Z$. In that limit eq. (75) becomes

$$\tilde{Z}(\omega) \simeq -iZ \tan(\omega\ell/c) + \tilde{Z}_\ell \sec^2(\omega\ell/c) . \quad (76)$$

In the limit of low frequency, $\omega\ell/c \ll 1$ this becomes

$$\tilde{Z}(\omega) = -i\omega \frac{Z\ell}{c} + \tilde{Z}_\ell = -i\omega \ell \mathcal{L} + \tilde{Z}_\ell , \quad (77)$$

using $\mathcal{L} = Z/c$. The final expression is the impedance of the load *in series* with an inductor $\ell \mathcal{L}$. This is exactly what we found in sec 1.5.

For the opposite case, $\tilde{Z}_\ell \ll Z$, we expand the inverse

$$\frac{1}{\tilde{Z}(\omega)} = \frac{1}{Z} \left[\frac{\tan(\omega\ell/c) + iZ/\tilde{Z}_\ell}{i + (Z/\tilde{Z}_\ell) \tan(\omega\ell/c)} \right] \simeq -i \frac{\tan(\omega\ell/c)}{Z} + \frac{\sec^2(\omega\ell/c)}{\tilde{Z}_\ell} . \quad (78)$$

Taking the low-frequency limit, $\omega\ell/c \ll 1$, of this case and using $cZ = 1/\mathcal{C}$, gives

$$\frac{1}{\tilde{Z}(\omega)} \simeq -i\omega\ell\mathcal{C} + \frac{1}{\tilde{Z}_\ell} . \quad (79)$$

This shows that the load is *in parallel* with a capacitor $\ell\mathcal{C}$.

Green's function

The circuit impedance in eq. (75) can be used to find the Green's function from which every response can be computed. If the transmission line is subject to a voltage impulse, $V(0, t) = V_0\delta(t)$, the current at that position is found from the inverse Fourier transform

$$I(0, t) = \frac{V_0}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\tilde{Z}(\omega)} d\omega . \quad (80)$$

The circuit impedance has zeros, and thus the integrand has poles, at frequencies satisfying

$$i \tan(\omega\ell/c) = \tanh(i\omega\ell/c) = \frac{\tilde{Z}_\ell}{Z} . \quad (81)$$

Because tan is periodic, there are an infinite number of such frequencies. These appear as an infinite number of normal mode frequencies, giving the transmission line a response far more complicated than simple circuit elements.

For a purely resistive load, $\tilde{Z}_\ell = R < Z$, the infinite number of frequencies lie along a horizontal line below the real axis

$$\omega_n = \pi n c / \ell - i\gamma = \frac{2\pi}{T} n - i\gamma \quad , \quad n = 0, \pm 1, \pm 2 \dots \quad (82)$$

where $T = 2\ell/c$ is the round-trip transit time and γ satisfies

$$\tanh(\gamma\ell/c) = \tanh(\gamma T/2) = \frac{R}{Z} . \quad (83)$$

Each of these frequencies is a simple zero of the circuit impedance

$$\tilde{Z}(\omega) \simeq -iZ \frac{\ell}{c} (\omega - \omega_n) = -\frac{1}{2}iZT(\omega - \omega_n) , \quad (84)$$

which is a simple pole in the integrand of eq. (80)

For positive times, $t > 0$, the integral (80) can be closed in the lower half of the complex plane. Doing so encloses all the poles, and the integral can be found by summing $-2\pi i$ times the residues

$$I(0, t) = \frac{2V_0}{ZT} e^{-\gamma t} \sum_{n=-\infty}^{\infty} e^{-2\pi i n t / T} , \quad t > 0 . \quad (85)$$

The final sum is a Fourier series of a chain of δ functions

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i n t/T} = T \sum_{j=-\infty}^{\infty} \delta(t - jT) , \quad (86)$$

so the current is

$$I(0, t) = \frac{V_0}{Z} \left[\delta(t) + 2 \sum_{j=1}^{\infty} e^{-\gamma j T} \delta(t - jT) \right] , \quad (87)$$

where we have restricted the sum to $t > 0$ but added the $\delta(t)$ to account for the current at the initial impulse. The response is a chain of successively decaying impulses from reflections off the load. To complete the analysis we use (83) to write

$$e^{-\gamma T} = \frac{1 - \tanh(\gamma T/2)}{1 + \tanh(\gamma T/2)} = \frac{Z - R}{Z + R} = \alpha , \quad (88)$$

where α was the factor used in sec. 1.5. In terms of this the current is

$$I(0, t) = \frac{V_0}{Z} \left[\delta(t) + \sum_{n=1}^{\infty} 2\alpha^n \delta(t - nT) \right] . \quad (89)$$

The response to a step function input, $V(0, t) = V_0 \Theta(t)$ is found from the convolution with this, which is just its integral from $-\infty$ to t . The result is a series of steps, exactly as found in sec. 1.5.

Differential approach to circuit inductance

As a final exercise we create a transmission line by connecting together infinitesimal lumped circuit elements. To an assembled line with circuit impedance $\tilde{Z}(\ell)$ we add a small segment with length $\delta\ell \ll \ell$. This segment has capacitance $\delta C = \mathcal{C} \delta\ell$ and inductance $\delta L = \mathcal{L} \delta\ell$, which add to the existing impedance to create $\tilde{Z}(\ell + \delta\ell)$, as shown in fig. 5. The inductance is added in series, and the capacitance in parallel with that combination. The reciprocal of the new circuit impedance is the sum of the reciprocal impedances of the parallel components

$$\frac{1}{\tilde{Z}(\ell + \delta\ell)} = \frac{1}{\tilde{Z}(\ell) - i\omega\delta L} - i\omega\delta C \simeq \frac{1}{\tilde{Z}(\ell)} + \left[\frac{i\omega\mathcal{L}}{\tilde{Z}^2(\ell)} - i\omega\mathcal{C} \right] \delta\ell , \quad (90)$$

after expanding in $|\omega\delta L/\tilde{Z}| \ll 1$. Expanding the left hand side

$$\frac{1}{\tilde{Z}(\ell + \delta\ell)} = \frac{1}{\tilde{Z}(\ell)} - \frac{1}{\tilde{Z}^2(\ell)} \frac{d\tilde{Z}}{d\ell} \delta\ell + \dots , \quad (91)$$

and equating the expressions term-by-term leads to the differential equation

$$\frac{d\tilde{Z}}{d\ell} = i\omega\mathcal{C} \tilde{Z}^2 - i\omega\mathcal{L} = i\omega\mathcal{C} \left[\tilde{Z}^2 - Z^2 \right] , \quad (92)$$

after using $Z^2 = \mathcal{L}/\mathcal{C}$. The expression in eq. (75) is the solution to this equation with initial condition $\tilde{Z}(\ell = 0) = \tilde{Z}_\ell$.

This exercise has shown that a transmission line can be envisioned as a ladder of capacitors and inductors. Furthermore, we can obtain the asymptotic impedance of a transmission line by taking the limit $\ell \rightarrow \infty$ in expression (92). The result is then one that makes $d\tilde{Z}/d\ell = 0$, which is evidently $\tilde{Z} = Z = \sqrt{\mathcal{L}/\mathcal{C}}$.

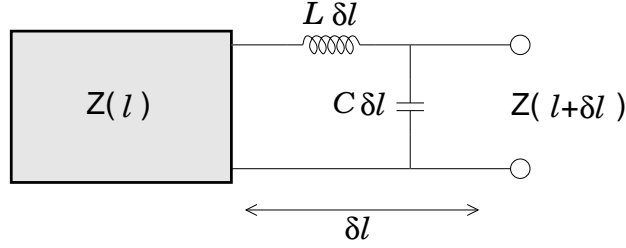


Figure 5: A step in the computation of the impedance of a transmission line. To a partial line, with impedance $\tilde{Z}(\ell)$, is added an inductance, $\mathcal{L}\delta\ell$ in series, and a capacitance, $\mathcal{C}\delta\ell$ in parallel. This extends the transmission line by $\delta\ell$, so the combination has impedance $\tilde{Z}(\ell + \delta\ell)$.

1.7 Transmission line with losses

The foregoing has dealt only with the ideal case of a transmission line with no losses. The most general extension includes resistive contributions of two different kinds. This first is the resistance present in the inner conductor. For simplicity we continue to assume the outer conductor is ideal, but it would be possible, with a bit more effort, to include its resistance into this same term. This resistance is accounted for by setting the electric field at the inner conductor, given by eq. (20), to

$$E_z(a, z, t) = -\frac{\partial\Phi(a, z, t)}{\partial z} - \frac{\partial A_z(a, z, t)}{\partial t} = \mathcal{R}I(a, z, t) , \quad (93)$$

where \mathcal{R} is the resistance per unit length of the inner conductor. Using eqs. (8) and (16) to replace $\Phi(a, z, t)$, and $A_z(a, z, t)$, gives

$$\frac{\partial I}{\partial t} + \frac{\mathcal{R}}{\mathcal{L}}I = -\frac{1}{\mathcal{L}}\frac{\partial V}{\partial z} , \quad (94)$$

in place of eq. (25). If the voltage and current were uniformly distributed, the current would therefore decay exponentially with time constant, \mathcal{L}/\mathcal{R} , exactly like a simple LR circuit.

The second contribution occurs if the material separating the conductors is not a perfect insulator, but has some resistance. This leads to an effect known as *leakage* between the conductors, quantified by *conductance* per unit length,⁵ typically denoted \mathcal{G} with units $\Omega^{-1}\text{m}^{-1}$. Leakage enters the equation of charge conservation, formerly expressed by eq. (23). The net charge on a section of inner conductor of length δz is $\delta Q = \lambda(a, z, t)\delta z$. This will change due to current entering the section from the left, and leaving it from the right or radially outward

$$\begin{aligned} \frac{d}{dt}\delta Q &= I(a, z, t) - I(a, z + \delta z, t) - \overbrace{(\mathcal{G}\delta z)V(a, z, t)}^{I_\rho} \\ &= \delta z \frac{\partial\lambda(a, z, t)}{\partial t} \simeq -\left[\frac{\partial I(a, z, t)}{\partial z} + \mathcal{G}V(a, z, t) \right] \delta z , \end{aligned}$$

where the radial current, I_ρ , is driven by the voltage difference between the conductors across a section of leaky insulator with conductance $\mathcal{G}\delta z$. Using eq. (9) to write $\lambda = \mathcal{C}V$ gives a revised

⁵Conductance is the reciprocal of resistance. When resistors are combined in parallel their conductances are summed to obtain the conductance of the combination. Annular sections of leaky insulator combine in *parallel* so their conductances add. Sections of inner conductor, on the other hand, add in series, so their resistances, $\mathcal{R}\delta z$, are summed in that case.

version of (26)

$$\frac{\partial V}{\partial t} + \frac{\mathcal{G}}{\mathcal{C}}V = -\frac{1}{\mathcal{C}}\frac{\partial I}{\partial z} . \quad (95)$$

In this case, a uniformly distributed current and voltage, the voltage would decay with a time constant, \mathcal{C}/\mathcal{G} , as it would for a traditional RC circuit — recall that conductance is the inverse of resistance, and that both \mathcal{C} and \mathcal{G} are defined per unit length.

Equations (94) and (95) are the lossy telegraphers equations, replacing the lossless versions, (25) and (26). They are still a set of coupled, first-order, linear PDEs, but they are a bit more complicated. They cannot, for example, be combined into the classical wave equation. To appreciate the complexity consider the wave solutions, $V(z, t) = \hat{V}e^{ikz-i\omega t}$ and $I(z, t) = \hat{I}e^{ikz-i\omega t}$. Placing these into eqs. (94) and (95) gives

$$\begin{bmatrix} \omega + i(\mathcal{R}/\mathcal{L}) & -k/\mathcal{L} \\ -k/\mathcal{C} & \omega + i(\mathcal{G}/\mathcal{C}) \end{bmatrix} \cdot \begin{bmatrix} \hat{I} \\ \hat{V} \end{bmatrix} = 0 . \quad (96)$$

A nontrivial solution requires the determinant vanish, yielding the dispersion relation

$$\omega^2 + i\left(\frac{\mathcal{R}}{\mathcal{L}} + \frac{\mathcal{G}}{\mathcal{C}}\right)\omega - \frac{\mathcal{R}\mathcal{G} + k^2}{\mathcal{L}\mathcal{C}} = 0 . \quad (97)$$

The classic dispersionless version, with $\omega = \pm k/\sqrt{\mathcal{L}\mathcal{C}}$, is recovered with perfect conductors, $\mathcal{R} = 0$, separated by a perfectly *insulating* material, $\mathcal{G} = 0$. Otherwise eq. (97) is more complicated than a traditional wave equation.

To appreciate this more complicated dispersion relation we compute the evolution of current or voltage initially distributed as $\propto \delta(z)$. The response to this initial condition will be proportional to

$$G(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz-i\omega(k)t} dk , \quad (98)$$

where $\omega(k)$ is the solution of eq. (97). Using the quadratic equation to solve eq. (97) gives the explicit dispersion relation

$$\omega(k) = -i\gamma \pm c\sqrt{k^2 - \kappa^2} , \quad (99)$$

where we have retained the original phase speed

$$c = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} , \quad (100)$$

and introduced the damping rate

$$\gamma = \frac{1}{2}\left(\frac{\mathcal{R}}{\mathcal{L}} + \frac{\mathcal{G}}{\mathcal{C}}\right) . \quad (101)$$

Both forms of loss contribute a negative imaginary part to the frequency, which produces exponential decay in the response — the expected effect of losses. The final parameter in the dispersion relation

$$\kappa = \frac{1}{2}c\left|\mathcal{R}\mathcal{C} - \mathcal{G}\mathcal{L}\right| , \quad (102)$$

is an inverse length characteristic of the losses. The upper and lower signs in eq. (99) are for rightward and leftward propagation, respectively, and the branch of the square root is followed continuously from $k > \kappa$ where it is positive to $k < -\kappa$, where it is negative.

Placing eq. (99) into eq. (98) gives an explicit form of the Green's function

$$G(z, t) = \frac{e^{-\gamma t}}{2\pi} \int_{-\infty}^{\infty} e^{ikz} \exp(\mp ict\sqrt{k^2 - \kappa^2}) dk , \quad (103)$$

where the upper and lower signs continue to apply to rightward and leftward propagation. The remaining integral is a formidable inverse Fourier transform which will produce a complicated function of space and time. This function is multiplied by an exponential damping as a result of the losses. But the losses also produce dispersion which renders the response far more complicated than that of a wave equation.

There is, however, one case where simple propagation is recovered. If resistance and leakage are perfectly balanced so that

$$\mathcal{R}\mathcal{C} = \mathcal{G}\mathcal{L} , \quad (104)$$

then $\kappa = 0$, according to eq. (102). In this case the integral in eq. (103) becomes a Dirac δ -function, leading to a Green's function

$$G(z, t) = \frac{e^{-\gamma t}}{2\pi} \int_{-\infty}^{\infty} e^{ik(z \mp ct)} dk = e^{-\gamma t} \delta(z \mp ct) = e^{-|z|/\ell_a} \delta(z \mp ct) , \quad (105)$$

where the attenuation length is

$$\ell_a = \frac{c}{\gamma} = \frac{Z}{\mathcal{R}} , \quad (106)$$

with impedance defined by $Z = \sqrt{\mathcal{L}/\mathcal{C}}$, as before. The initial pulse thus propagates along the lossy transmission line attenuated but undistorted. This Green's function can be used to find the evolution of an arbitrary initial distribution of current and voltage. Provided this distribution varies on scales much smaller than the attenuation length, ℓ_a , it will propagate with damping but no distortion. According to eq. (106), the attenuation length is the distance over which the accumulated resistance matches the cable's impedance. A good copper wire will require kilometers to match a typical impedance of say 50Ω . If, however, the transmission line fails to satisfy condition (104) the Green's function will not be of the form (105), and the shapes will be distorted as they propagate.

The condition for undistorted wave propagation, namely eq. (104), was first derived by Oliver Heaviside in the context of undersea telegraph cables and quickly applied to the newly invented telephone. Heaviside's insight was that, since it is impossible to completely eliminate resistance \mathcal{R} in any real conductor, eliminating distortion requires at least some leakage. If that is small, then one must also find a way to increase the inductance per unit length \mathcal{L} in order to satisfy the distortion-free requirement, eq. (104).

Circuit impedance of a lossy transmission line

The previous section computed the circuit impedance of a lossless transmission line in two different ways. The first used the analytic solution to the wave equation. We have no analogous solution for the lossy version, so that method cannot be applied in this case. This leaves us with the differential method where we add an infinitesimal section, $\delta\ell$ to an existing transmission line with circuit impedance $\tilde{Z}(\ell)$. The additional segment contributes, in parallel, a capacitor $\mathcal{C}\delta\ell$, a leakage

resistor with conductance $\mathcal{G}\delta\ell$, and the series combination of the resistor, $\mathcal{R}\delta\ell$, inductor, $\mathcal{L}\delta\ell$, and the existing the impedance, as shown in fig. 6. Summing these three reciprocal impedances of these parallel contributions gives the reciprocal impedance of the combination

$$\frac{1}{\tilde{Z}(\ell + \delta\ell)} = \frac{1}{\tilde{Z}(\ell) + \mathcal{R}\delta\ell - i\omega\mathcal{L}\delta\ell} + \mathcal{G}\delta\ell - i\omega\mathcal{C}\delta\ell . \quad (107)$$

Expanding all expressions to first order in the small segment, $\delta\ell$,

$$\frac{1}{\tilde{Z}(\ell)} - \frac{1}{\tilde{Z}^2(\ell)} \frac{d\tilde{Z}}{d\ell} \delta\ell = \frac{1}{\tilde{Z}(\ell)} - \left[\frac{\mathcal{R} - i\omega\mathcal{L}}{\tilde{Z}^2(\ell)} - \mathcal{G} + i\omega\mathcal{C} \right] \delta\ell , \quad (108)$$

and then equating the linear terms gives the differential equation

$$\frac{d\tilde{Z}}{d\ell} = \mathcal{R} - i\omega\mathcal{L} + (i\omega\mathcal{C} - \mathcal{G}) \tilde{Z}^2(\ell) . \quad (109)$$

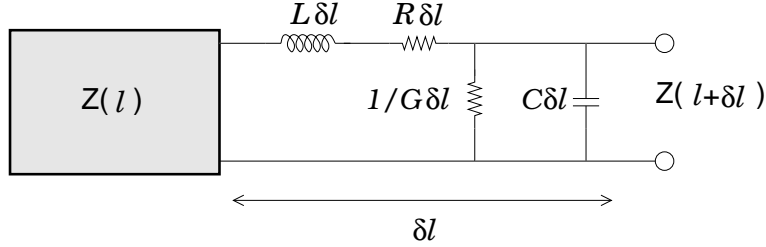


Figure 6: A step in the computation of the impedance of a lossy transmission line. To a partial line, with impedance $\tilde{Z}(\ell)$, is added an inductance, $\mathcal{L}\delta\ell$, and resistance $\mathcal{R}\delta\ell$, both in series, and a leakage conductance, $\mathcal{G}\delta\ell$ (resistance $1/\mathcal{G}\delta\ell$), and capacitance, $\mathcal{C}\delta\ell$, both in parallel. This extends the transmission line by $\delta\ell$, so the combination has impedance $\tilde{Z}(\ell + \delta\ell)$.

Equation (109) characterizes the circuit impedance of a lossy transmission line in analogy to eq. (92) for the lossless case. For clarity we can re-write the equation

$$\frac{d\tilde{Z}}{d\ell} = (i\omega\mathcal{C} - \mathcal{G}) \left[\tilde{Z}^2(\ell) - \tilde{Z}_\infty^2 \right] , \quad (110)$$

after introducing the asymptotic impedance

$$\tilde{Z}_\infty(\omega) = \sqrt{\frac{i\omega\mathcal{L} - \mathcal{R}}{i\omega\mathcal{C} - \mathcal{G}}} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \times \sqrt{\frac{\omega + i\mathcal{R}/\mathcal{L}}{\omega + i\mathcal{G}/\mathcal{C}}} , \quad (111)$$

for a lossy transmission line of infinite length. The pre-factor in the final expression is the characteristic impedance $Z = \sqrt{\mathcal{L}/\mathcal{C}}$. The lossy version approaches this at frequencies above the characteristic frequencies of the losses, \mathcal{R}/\mathcal{L} and \mathcal{G}/\mathcal{C} . Furthermore, under the condition for no distortion, eq. (104), these frequencies are identical, the second factor becomes unity and $\tilde{Z}_\infty = Z$.

To find the circuit impedance we re-write eq. (110) in the form

$$\begin{aligned} \left(\frac{1}{\tilde{Z} - \tilde{Z}_\infty} - \frac{1}{\tilde{Z} + \tilde{Z}_\infty} \right) \frac{d\tilde{Z}}{d\ell} &= \frac{d}{d\ell} \ln \left(\frac{\tilde{Z} - \tilde{Z}_\infty}{\tilde{Z} + \tilde{Z}_\infty} \right) \\ &= 2\tilde{Z}_\infty (i\omega\mathcal{C} - \mathcal{G}) = 2\sqrt{(i\omega\mathcal{L} - \mathcal{R})(i\omega\mathcal{C} - \mathcal{G})} . \end{aligned} \quad (112)$$

We re-write the rhs

$$2\sqrt{(i\omega\mathcal{L} - \mathcal{R})(i\omega\mathcal{C} - \mathcal{G})} = 2i\sqrt{\mathcal{L}\mathcal{C}}\sqrt{(\omega + i\mathcal{R}/\mathcal{L})(\omega + i\mathcal{G}/\mathcal{C})} = 2iw(\omega)/c, \quad (113)$$

where $c = 1/\sqrt{\mathcal{L}\mathcal{C}}$ is the signal speed as in the lossless case. This function of frequency goes over to $w(\omega) = \omega$ in the purely ideal case, $\mathcal{R} = \mathcal{G} = 0$. In the distortionless case, $\mathcal{R}/\mathcal{L} = \mathcal{G}/\mathcal{C} = \gamma$ it becomes $w(\omega) = \omega + i\gamma$.

The solution to eq. (112), for initial condition $\tilde{Z}(0) = \tilde{Z}_\ell$ is

$$\tilde{Z}(\ell) = \tilde{Z}_\infty \left\{ \frac{\tilde{Z}_\infty \tan[w(\omega)\ell/c] + i\tilde{Z}_\ell}{\tilde{Z}_\ell \tan[w(\omega)\ell/c] + i\tilde{Z}_\infty} \right\}. \quad (114)$$

This is obviously analogous to eq. (75) with the arguments of the tangents changed $\omega \rightarrow w(\omega)$ and $Z \rightarrow \tilde{Z}_\infty$.

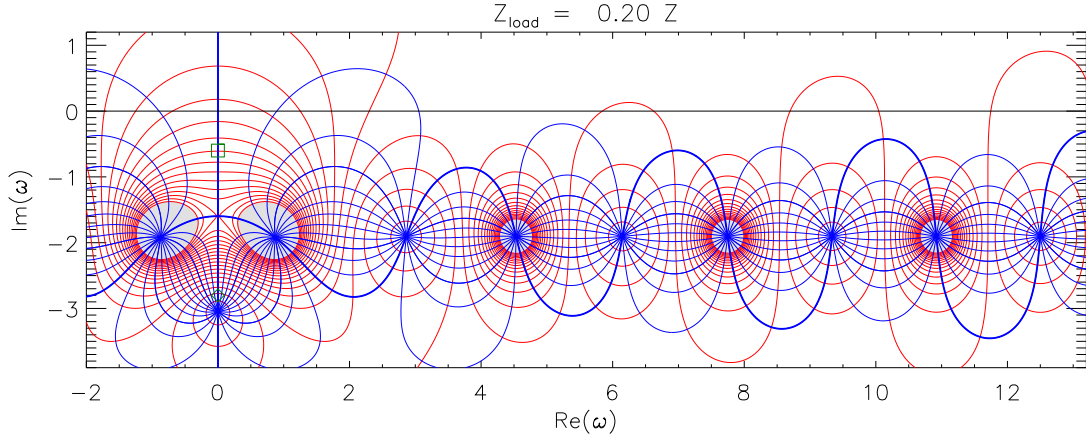


Figure 7: The complex impedance of a lossy transmission line plotted within the complex frequency plane in units of c/ℓ . Red and blue curves show the amplitude and argument (phase) of $\tilde{Z}(\omega)$ respectively. The impedance is purely real along the thick blue curve which includes the axis of imaginary frequency. Poles occur inside shaded regions. A green diamond and square show $-i\mathcal{R}/\mathcal{L}$ and $-i\mathcal{G}/\mathcal{C}$ respectively.

A separate derivation involves solving eqs. (94) and (95) for boundary condition $V(0, t) = e^{-i\omega t}$. With the specified time dependence they become

$$\left(-i\omega + \frac{\mathcal{R}}{\mathcal{L}}\right) I = -i\left(\omega + i\frac{\mathcal{R}}{\mathcal{L}}\right) I = -\frac{1}{\mathcal{L}} \frac{\partial V}{\partial z} \quad (115)$$

$$\left(-i\omega + \frac{\mathcal{G}}{\mathcal{C}}\right) V = -i\left(\frac{\omega + i\mathcal{G}}{\mathcal{C}}\right) V = -\frac{1}{\mathcal{C}} \frac{\partial I}{\partial z}. \quad (116)$$

They are combined to

$$\left(\frac{\mathcal{R}}{\mathcal{L}} - i\omega\right) \left(\frac{\mathcal{G}}{\mathcal{C}} - i\omega\right) V = -\overbrace{\left(\omega + i\frac{\mathcal{R}}{\mathcal{L}}\right) \left(\omega + i\frac{\mathcal{G}}{\mathcal{C}}\right)}^{w^2(\omega)} V = c^2 \frac{\partial^2 V}{\partial z^2}, \quad (117)$$

where the combination $w^2(\omega)$ is identified from above. The general solution matching the boundary condition is

$$V(z) = \cos[w(\omega)z/c] + \tilde{A} \sin[w(\omega)z/c] , \quad (118)$$

for an, as yet, unspecified value of \tilde{A} . The current is

$$\begin{aligned} I(z) &= -\frac{i}{\omega + i\mathcal{R}/\mathcal{L}} \left(\frac{1}{\mathcal{L}} \right) \frac{\partial V}{\partial z} = \frac{iw(\omega)/c\mathcal{L}}{\omega + i\mathcal{R}/\mathcal{L}} \left\{ \sin[w(\omega)z/c] - \tilde{A} \cos[w(\omega)z/c] \right\} \\ &= \frac{i}{\tilde{Z}_\infty(\omega)} \left\{ \sin[w(\omega)z/c] - \tilde{A} \cos[w(\omega)z/c] \right\} , \end{aligned} \quad (119)$$

after combining

$$\frac{w(\omega)/c\mathcal{L}}{\omega + i\mathcal{R}/\mathcal{L}} = \sqrt{\frac{\omega + i\mathcal{G}/\mathcal{C}}{\omega + i\mathcal{R}/\mathcal{L}}} \sqrt{\frac{\mathcal{C}}{\mathcal{L}}} = \frac{1}{\tilde{Z}_\infty(\omega)} \quad (120)$$

Using expression (119) to find $\tilde{Z}_\ell I(\ell)$ and equating that with $V(\ell)$ yields

$$i \frac{\tilde{Z}_\ell}{\tilde{Z}_\infty} \left\{ \sin[w(\omega)\ell/c] - \tilde{A} \cos[w(\omega)\ell/c] \right\} = \cos[w(\omega)\ell/c] + \tilde{A} \sin[w(\omega)\ell/c] . \quad (121)$$

Multiplying by \tilde{Z}_∞ and dividing by the cosine gives the expression

$$i\tilde{Z}_\ell \left\{ \tan[w(\omega)\ell/c] - \tilde{A} \right\} = \tilde{Z}_\infty \left\{ 1 + \tilde{A} \tan[w(\omega)\ell/c] \right\} . \quad (122)$$

Solving for \tilde{A} yields

$$\tilde{A} = \frac{i\tilde{Z}_\ell \tan[w(\omega)\ell/c] - \tilde{Z}_\infty}{i\tilde{Z}_\ell + \tilde{Z}_\infty \tan[w(\omega)\ell/c]} \quad (123)$$

We evaluate eq. (119) at $z = 0$ to obtain

$$I(0) = \frac{1}{\tilde{Z}(\omega)} = -\frac{i}{\tilde{Z}_\infty(\omega)} \tilde{A} . \quad (124)$$

This gives the circuit impedance

$$\tilde{Z}(\omega) = -\frac{\tilde{Z}_\infty(\omega)}{i\tilde{A}} = \tilde{Z}_\infty(\omega) \left\{ \frac{\tilde{Z}_\infty \tan[w(\omega)\ell/c] + i\tilde{Z}_\ell}{\tilde{Z}_\ell \tan[w(\omega)\ell/c] + i\tilde{Z}_\infty} \right\} \quad (125)$$

This has the limit $\tilde{Z} \rightarrow \tilde{Z}_\ell$ when $\ell \rightarrow 0$; this is reassuring.

For purely imaginary frequency we write

$$\omega + i\frac{\mathcal{R}}{\mathcal{L}} = i\frac{c}{\ell} D_L \quad , \quad \omega + i\frac{\mathcal{G}}{\mathcal{C}} = i\frac{c}{\ell} D_C \quad , \quad (126)$$

and thus

$$\frac{w(\omega)\ell}{c} = i\sqrt{D_L D_C} \quad , \quad \tilde{Z}_\infty(\omega) = Z \sqrt{\frac{D_L}{D_C}} \quad , \quad (127)$$

where $Z = \sqrt{\mathcal{L}/\mathcal{C}}$, as always. The circuit impedance then becomes

$$\begin{aligned}\tilde{Z}(\omega) &= Z \sqrt{\frac{D_L}{D_C}} \left[\frac{Z \sqrt{D_L/D_C} \tanh(\sqrt{D_L D_C}) + \tilde{Z}_\ell}{\tilde{Z}_\ell \tanh(\sqrt{D_L D_C}) + Z \sqrt{D_L/D_C}} \right] \\ &= Z \left[\frac{Z \sqrt{D_L/D_C} \tanh(\sqrt{D_L D_C}) + \tilde{Z}_\ell}{\tilde{Z}_\ell \sqrt{D_C/D_L} \tanh(\sqrt{D_L D_C}) + Z} \right]\end{aligned}\quad (128)$$

When the line is shorted or open ($\tilde{Z}_\ell = 0$ or ∞ respectively), the impedance of the transmission line is

$$\tilde{Z}_{\text{short}} = Z \sqrt{\frac{D_L}{D_C}} \tanh(\sqrt{D_L D_C}) \quad , \quad \tilde{Z}_{\text{open}} = Z \sqrt{\frac{D_L}{D_C}} \coth(\sqrt{D_L D_C}) \quad . \quad (129)$$

The limit of zero frequency can be approached using this expression and taking

$$\sqrt{D_L D_C} = \frac{\ell}{c} \sqrt{\frac{\mathcal{R}\mathcal{G}}{\mathcal{L}\mathcal{C}}} = \ell \sqrt{\mathcal{R}\mathcal{G}} \quad , \quad Z \sqrt{\frac{D_L}{D_C}} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \sqrt{\frac{\mathcal{R}\mathcal{C}}{\mathcal{L}\mathcal{G}}} = \sqrt{\frac{\mathcal{R}}{\mathcal{G}}} \quad . \quad (130)$$

The impedance at zero frequency

$$\tilde{Z}(0) = \sqrt{\frac{\mathcal{R}}{\mathcal{G}}} \left[\frac{\tanh(\ell \sqrt{\mathcal{R}\mathcal{G}}) + (\tilde{Z}_\ell/Z)}{(\tilde{Z}_\ell/Z) \tanh(\ell \sqrt{\mathcal{R}\mathcal{G}}) + 1} \right] \quad , \quad (131)$$

is purely resistive, provided the load is resistive (\tilde{Z}_ℓ is real). In the limit $\ell \rightarrow \infty$ this gives $\tilde{Z}(0) \rightarrow \tilde{Z}_\infty(0) = \sqrt{\mathcal{R}/\mathcal{G}}$.

The bottom expression in eq. (128) is regular at points where D_L or D_C vanish — there are neither singularities nor branch points there. $\tanh()$ is an odd function with only odd powers of its argument, so the products in the numerator and denominator involve only non-negative whole powers of D_L and D_C . This means that the impedance is purely real along the entire axis of imaginary ω (i.e. real D_L and D_C).

In the limit where $D_C \rightarrow 0$, the product in the numerator of the bottom expression in eq. (128) goes to D_L , and in the denominator to $D_C \rightarrow 0$. The result is

$$\tilde{Z}(-i\sqrt{\mathcal{G}/\mathcal{C}}) \simeq Z D_L + \tilde{Z}_\ell \quad , \quad (132)$$

where $D_L = (\ell\mathcal{R}/Z) - (\ell\mathcal{G}Z)$ here. In the other limit, $D_L \rightarrow 0$, the product in the numerator goes to $D_L \rightarrow 0$, while the product in the denominator to D_C . The result is

$$\tilde{Z}(-i\sqrt{\mathcal{R}/\mathcal{L}}) \simeq \frac{Z \tilde{Z}_\ell}{Z D_C + \tilde{Z}_\ell} \quad , \quad (133)$$

where $D_C = (\ell\mathcal{G}Z) - (\ell\mathcal{R}/Z)$ here.

When both D_L and D_C are negative, we must choose the branches where $\sqrt{D_L D_C}$ is negative and $\sqrt{D_L/D_C}$ is positive. This means that the numerator of eq. (128) will vanish, and the impedance will have a zero, where

$$\sqrt{\frac{|D_L|}{|D_C|}} \tanh(\sqrt{|D_L D_C|}) = \frac{\tilde{Z}_\ell}{Z} \quad . \quad (134)$$

The denominator will vanish, and the impedance have a pole, where

$$\sqrt{\frac{|D_L|}{|D_C|}} \coth(\sqrt{|D_L D_C|}) = \frac{\tilde{Z}_\ell}{Z} . \quad (135)$$