

MHD Stability

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We have found all linear modes for homogeneous magnetostatic equilibrium: $p_0(\mathbf{x}) = p_0$ and $\mathbf{B}_0(\mathbf{x}) = B_0 \hat{\mathbf{z}}$. The linear modes are plane waves with real eigenfrequencies in 4 branches: entropy mode ($\omega = 0$), shear Alfvén waves $\omega \pm \mathbf{k} \cdot \hat{\mathbf{z}} B_0 / \sqrt{4\pi\rho}$, and fast and slow magnetosonic waves. The situation becomes more complicated when more complicated equilibria are considered: for example an equilibrium with magnetic field $\mathbf{B}_0(\mathbf{x})$ which is not uniform.

We consider next a more complicated case. We restrict the complexity by limiting ourselves to cases with $\beta \ll 1$. Equilibria in this case force-free magnetic fields

$$\nabla \times \mathbf{B}_0 = \alpha \mathbf{B}_0 \quad , \quad \mathbf{B}_0 \cdot \nabla \alpha = 0 \quad . \quad (\text{MS.1})$$

Due to the low β , the pressure term may be dropped from the momentum equation. It therefore becomes unnecessary to solve the energy equation. The perturbed mass density does not enter into any of the equations so it is also unnecessary to solve the continuity equation. Linearization of the remaining equations are

$$\begin{aligned} -i\omega \hat{\mathbf{u}}_1 &= \frac{1}{4\pi\rho_0(\mathbf{x})} (\nabla \times \mathbf{B}_0) \times \hat{\mathbf{B}}_1 + \frac{1}{4\pi\rho_0(\mathbf{x})} (\nabla \times \hat{\mathbf{B}}_1) \times \mathbf{B}_0 \\ &= \frac{\mathbf{B}_0(\mathbf{x})}{4\pi\rho_0(\mathbf{x})} \times \left[\alpha(\mathbf{x}) \hat{\mathbf{B}}_1 - \nabla \times \hat{\mathbf{B}}_1 \right] \end{aligned} \quad (\text{MS.2})$$

$$-i\omega \hat{\mathbf{B}}_1 = \nabla \times (\hat{\mathbf{u}}_1 \times \mathbf{B}_0) \quad , \quad (\text{MS.3})$$

after making the usual normal mode form

$$\mathbf{u}_1(\mathbf{x}, t) = \hat{\mathbf{u}}_1(\mathbf{x}) e^{-i\omega t} \quad , \quad \mathbf{B}_1(\mathbf{x}, t) = \hat{\mathbf{B}}_1(\mathbf{x}) e^{-i\omega t} \quad . \quad (\text{MS.4})$$

Without assuming any spatial symmetry in the equilibrium field, $\mathbf{B}_0(\mathbf{x})$, we cannot predict how the mode amplitudes, $\hat{\mathbf{u}}_1(\mathbf{x})$ and $\hat{\mathbf{B}}_1(\mathbf{x})$ will depend on space.

These two can be combined into a single vector equation for eigenvalue ω

$$\omega^2 \hat{\mathbf{u}}_1 = \frac{\mathbf{B}_0(\mathbf{x})}{4\pi\rho_0(\mathbf{x})} \times \left[\nabla \times \nabla \times (\hat{\mathbf{u}}_1 \times \mathbf{B}_0) - \alpha \nabla \times (\hat{\mathbf{u}}_1 \times \mathbf{B}_0) \right] = \mathbf{F}\{\hat{\mathbf{u}}_1\} \quad , \quad (\text{MS.5})$$

where \mathbf{F} is a vector-valued operator, linear in its argument, $\hat{\mathbf{u}}_1(\mathbf{x})$. The linearized equation is an eigenvalue equation with eigenvalue, ω^2 . The operator depends on the equilibrium magnetic field $\mathbf{B}_0(\mathbf{x})$ and can be quite complicated.

One set of eigenvalues can, however, be quickly identified. Posing an eigenfunction

$$\hat{\mathbf{u}}_1(\mathbf{x}) = \hat{\psi}(\mathbf{x}) \mathbf{B}_0(\mathbf{x}) \quad , \quad (\text{MS.6})$$

where $\hat{\psi}(\mathbf{x})$ is an arbitrary scalar function, leads to $\mathbf{F}\{\hat{\mathbf{u}}_1(\mathbf{x})\} = 0$ and thus $\omega = 0$. This infinite variety of zero-frequency eigenmodes correspond to modes related to the slow magnetosonic waves, which at $\beta \rightarrow 0$ have zero frequency. They are not plane waves, but do comprise an infinite variety in as much as the function $\hat{\psi}(\mathbf{x})$ is arbitrary.

The Eikenol approximation offers second approach to finding eigenfuncitons of \mathbf{F} . We pose eigenfunctions of the form

$$\hat{\mathbf{u}}_1(\mathbf{x}) = \hat{\mathbf{A}}(\mathbf{x}) e^{i\phi(x)} \quad , \quad (\text{MS.7})$$

and assume $|\nabla\phi| \gg$ than all other inverse length scales. Only spatial derivatives acting on the complex phase factor will be large enough to be significant. We may therefore replace every $\nabla \rightarrow \nabla\phi = i\mathbf{k}$. We may also discard the second term in \mathbf{F} since it involves only a single spatial derivative, compare to the two in the first term. The result is

$$\begin{aligned}
\mathbf{F}(\hat{\mathbf{u}}_1) &= -\frac{\mathbf{B}_0}{4\pi\rho_0} \times \left\{ \mathbf{k} \times \left[\mathbf{k} \times (\hat{\mathbf{u}}_1 \times \mathbf{B}_0) \right] \right\} \\
&= -\frac{\mathbf{B}_0}{4\pi\rho_0} \times \left\{ \mathbf{k} \times \left[(\mathbf{k} \cdot \mathbf{B}_0)\hat{\mathbf{u}}_1 - (\mathbf{k} \cdot \hat{\mathbf{u}}_1)\mathbf{B}_0 \right] \right\} \\
&= -\frac{1}{4\pi\rho_0} \left\{ [\mathbf{B}_0 \times (\mathbf{k} \times \hat{\mathbf{u}}_1)](\mathbf{k} \cdot \mathbf{B}_0) - [\mathbf{B}_0 \times (\mathbf{k} \times \mathbf{B}_0)](\mathbf{k} \cdot \hat{\mathbf{u}}_1) \right\} \\
&= \frac{1}{4\pi\rho_0} \left\{ (\mathbf{k} \cdot \mathbf{B}_0)^2 \hat{\mathbf{u}}_1 - (\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{B}_0 \cdot \hat{\mathbf{u}}_1)\mathbf{k} + B_0^2(\mathbf{k} \cdot \hat{\mathbf{u}}_1)\mathbf{k} - (\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{k} \cdot \hat{\mathbf{u}}_1)\mathbf{B}_0 \right\} \\
&= \frac{1}{4\pi\rho_0} \left\{ (\mathbf{k} \cdot \mathbf{B}_0)^2 \hat{\mathbf{u}}_1 + B_0^2(\mathbf{k} \cdot \hat{\mathbf{u}}_1)\mathbf{k} - (\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{k} \cdot \hat{\mathbf{u}}_1)\mathbf{B}_0 \right\} , \tag{MS.8}
\end{aligned}$$

after using the fact that $\mathbf{B}_0 \cdot \hat{\mathbf{u}}_1 = 0$ once we discard the slow magnetosonic modes. This can be contracted with \mathbf{k} to yield the equation

$$\mathbf{k} \cdot \mathbf{F}(\hat{\mathbf{u}}_1) = \frac{B_0^2 k^2}{4\pi\rho_0} (\mathbf{k} \cdot \hat{\mathbf{u}}_1) = \omega^2 (\mathbf{k} \cdot \hat{\mathbf{u}}_1) , \tag{MS.9}$$

for the compressive wave mode. This leads to the eigenfrequencies

$$\omega(\mathbf{k}) = \pm v_A k , \tag{MS.10}$$

which are fast magnetosonic waves in the $\beta = 0$ limit; the dispersion relationship is isotropic.

Contracting (MS.8) with the orthogonal vector, $\mathbf{k} \times \mathbf{B}_0$ leads to

$$(\mathbf{k} \times \mathbf{B}_0) \cdot \mathbf{F}(\hat{\mathbf{u}}_1) = \frac{1}{4\pi\rho_0} (\mathbf{k} \cdot \mathbf{B}_0)^2 (\mathbf{k} \times \mathbf{B}_0 \cdot \hat{\mathbf{u}}_1) = \omega^2 (\mathbf{k} \times \mathbf{B}_0 \cdot \hat{\mathbf{u}}_1) . \tag{MS.11}$$

The eigenvalues from this

$$\omega(\mathbf{k}) = \pm \frac{\mathbf{k} \cdot \mathbf{B}_0}{\sqrt{4\pi\rho_0}} = \pm v_A \hat{\mathbf{b}}_0 \cdot \mathbf{k} , \tag{MS.12}$$

are shear Alfvén waves; $\hat{\mathbf{b}}_0 = \mathbf{B}_0/B_0$ is the unit vector along the local magnetic field.

This illustrates several solutions to the eigenvalue equation (MS.5). They are related to the modes of a homogeneous equilibrium. That relation is inevitable in the Eikonal limit, since it demands modes whose scales are so small than any equilibrium “appears” homogeneous. The more interesting cases are those where the eigenmode, $\hat{\mathbf{u}}_1(\mathbf{x})$ varies on spatial scales comparable to those in the equilibrium $\mathbf{B}_0(\mathbf{x})$. In this case things are far more complicated.

MS.1 Properties of the linear operator

According to eq. (MS.5) the square of the eigenfrequency, ω^2 , is an eigenvalue of the vector-valued linear operator \mathbf{F} . This operator is fairly complicated, especially since it depends

on the equilibrium magnetic field $\mathbf{B}_0(\mathbf{x})$, which could be very complicated. The operator is *self-adjoint* with respect to a particular inner product for complex vector-valued functions, $\hat{\mathbf{u}}(\mathbf{x})$ and $\hat{\mathbf{w}}(\mathbf{x})$,

$$\langle \hat{\mathbf{w}} | \hat{\mathbf{u}} \rangle = \int_{\mathcal{V}} \rho_0(\mathbf{x}) \hat{\mathbf{w}}^*(\mathbf{x}) \cdot \hat{\mathbf{u}}(\mathbf{x}) d^3x \quad , \quad (\text{MS.13})$$

where the integral is over the volume \mathcal{V} occupied by the equilibrium. This inner product is a scalar, linear in each of its arguments and satisfies requirements of an inner product

$$\langle \hat{\mathbf{w}} | \hat{\mathbf{u}} \rangle = \langle \hat{\mathbf{u}} | \hat{\mathbf{w}} \rangle^* \quad , \quad \langle \hat{\mathbf{u}} | \hat{\mathbf{u}} \rangle \geq 0 \quad .$$

It depends on the plasma density, $\rho_0(\mathbf{x})$, which we must assume to be finite and non-zero everywhere within the domain \mathcal{V} .

The property of self-adjointness depends on the inner product and also on boundary conditions satisfied by the functions. We will assume that the plasma velocity vanishes at the boundary of this volume: $\hat{\mathbf{u}} = \hat{\mathbf{w}} = 0$ on $\partial\mathcal{V}$. The inner product of one function $\hat{\mathbf{w}}(\mathbf{x})$, with the linear operator from eq. (MS.5) operating on a second, $\hat{\mathbf{u}}(\mathbf{x})$, can be rewritten

$$\begin{aligned} \langle \hat{\mathbf{w}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle &= \frac{1}{4\pi} \int_{\mathcal{V}} \hat{\mathbf{w}}^* \cdot \left\{ \mathbf{B}_0 \times \left[\nabla \times \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) - \alpha \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right] \right\} d^3x \\ &= \frac{1}{4\pi} \int_{\mathcal{V}} (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\ &\quad - \frac{1}{4\pi} \int_{\mathcal{V}} \alpha(\mathbf{x}) (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\ &= -\frac{1}{4\pi} \oint_{\partial\mathcal{V}} (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \times \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \cdot d\mathbf{a} \\ &\quad + \frac{1}{4\pi} \int_{\mathcal{V}} \nabla \times (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\ &\quad - \frac{1}{4\pi} \int_{\mathcal{V}} \alpha(\mathbf{x}) (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\ &= \frac{1}{4\pi} \int_{\mathcal{V}} \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\ &\quad - \frac{1}{4\pi} \int_{\mathcal{V}} \alpha(\mathbf{x}) (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \end{aligned}$$

after invoking $\hat{\mathbf{w}} = 0$ on $\partial\mathcal{V}$ to eliminate surface terms.¹ We have also used the fact that the equilibrium magnetic field $\mathbf{B}_0(\mathbf{x})$ is real. The second term can be rewritten using an integration by parts

$$\int_{\mathcal{V}} \alpha (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x = \frac{1}{2} \int_{\mathcal{V}} \alpha (\hat{\mathbf{w}}^* \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x$$

¹In fact only the normal component of the displacement need vanish, $\hat{\mathbf{n}} \cdot \hat{\mathbf{w}} = 0$ as long as $\hat{\mathbf{n}} \cdot \mathbf{B}_0 = 0$ on the surface.

$$\begin{aligned}
& -\frac{1}{2} \oint_{\partial \mathcal{V}} \alpha \left[(\hat{\mathbf{w}}^* \times \mathbf{B}_0) \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right] \cdot d\mathbf{a} + \frac{1}{2} \int_{\mathcal{V}} \nabla \times [\alpha (\hat{\mathbf{w}}^* \times \mathbf{B}_0)] \cdot (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \\
& = \frac{1}{2} \int_{\mathcal{V}} \alpha \left[(\hat{\mathbf{w}} \times \mathbf{B}_0) \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0)^* + \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot (\hat{\mathbf{u}} \times \mathbf{B}_0) \right] d^3x \\
& - \frac{1}{2} \int_{\mathcal{V}} \nabla \alpha \cdot (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \times (\hat{\mathbf{u}} \times \mathbf{B}_0) d^3x \quad ,
\end{aligned}$$

after the dropping surface term once again. The inner product is therefore

$$\begin{aligned}
\langle \hat{\mathbf{w}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle & = \frac{1}{4\pi} \int_{\mathcal{V}} \left[\nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) + \frac{1}{2} \nabla \alpha \cdot (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right] d^3x \\
& - \frac{1}{8\pi} \int_{\mathcal{V}} \alpha(\mathbf{x}) \left[(\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) + \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot (\hat{\mathbf{u}} \times \mathbf{B}_0) \right] d^3x \quad (\text{MS.14})
\end{aligned}$$

The steps above shows that the inner product $\langle \hat{\mathbf{w}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle$ is symmetric in the functions $\hat{\mathbf{u}}(\mathbf{x})$ and $\hat{\mathbf{w}}^*(\mathbf{x})$. This leads immediately to the conclusion that \mathbf{F} is a self-adjoint operator

$$\langle \hat{\mathbf{w}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle = \langle \mathbf{F}(\hat{\mathbf{w}}) | \hat{\mathbf{u}} \rangle = \langle \hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{w}}) \rangle^* \quad , \quad (\text{MS.15})$$

which is the defining property of a Sturm-Liouville equation; evidently (MS.5) is a Sturm-Liouville equation, albeit a vector-valued version. Numerous consequences follow from this fact. The first and most significant is that eigenvalues of \mathbf{F} , namely ω^2 , must be real. Taking an inner product of (MS.5), shows that it corresponds to the ratio

$$\omega^2 = \frac{\langle \hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle}{\langle \hat{\mathbf{u}} | \hat{\mathbf{u}} \rangle} \quad , \quad (\text{MS.16})$$

which can be seen to be real, owing to the self-adjointness of \mathbf{F} . This does not, however, mean that the eigenfrequency ω is real, since

$$\langle \hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle = \frac{1}{4\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right|^2 - \alpha(\mathbf{x}) (\hat{\mathbf{u}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right\} d^3x \quad . \quad (\text{MS.17})$$

could be negative. Since its square is real any eigenvalue ω will be either purely real or purely imaginary.

Expression (MS.17), related to the eigenfrequency, is a quadratic form which we designate

$$W\{\hat{\mathbf{u}}\} = \frac{1}{2} \langle \hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle = \frac{1}{8\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right|^2 - \alpha(\mathbf{x}) (\hat{\mathbf{u}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{u}} \times \mathbf{B}_0) \right\} d^3x \quad . \quad (\text{MS.18})$$

Consider the variation of this function subject to the normalization constraint, $\langle \hat{\mathbf{u}} | \hat{\mathbf{u}} \rangle = 1$, applied by a Lagrange undetermined multiplier

$$\begin{aligned}
& W\{\hat{\mathbf{u}} + \delta\hat{\mathbf{u}}\} + \lambda \left[\langle \hat{\mathbf{u}} + \delta\hat{\mathbf{u}} | \hat{\mathbf{u}} + \delta\hat{\mathbf{u}} \rangle - 1 \right] \\
& = W\{\hat{\mathbf{u}}\} + \frac{1}{2} \langle \delta\hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle + \frac{1}{2} \langle \hat{\mathbf{u}} | \mathbf{F}(\delta\hat{\mathbf{u}}) \rangle + \lambda \langle \delta\hat{\mathbf{u}} | \hat{\mathbf{u}} \rangle + \lambda \langle \hat{\mathbf{u}} | \delta\hat{\mathbf{u}} \rangle + \dots \\
& = W\{\hat{\mathbf{u}}\} + \underbrace{\frac{1}{2} \langle \delta\hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) + 2\lambda \hat{\mathbf{u}} \rangle + \frac{1}{2} \langle \mathbf{F}(\hat{\mathbf{u}}) + 2\lambda \hat{\mathbf{u}} | \delta\hat{\mathbf{u}} \rangle}_{\delta W} + \dots \quad (\text{MS.19})
\end{aligned}$$

For $\hat{\mathbf{u}}(\mathbf{x})$ to be an extremum of W it is necessary that δW vanish for every variation $\delta\hat{\mathbf{u}}(\mathbf{x})$. For both the real and imaginary part of δW to vanish it is necessary that

$$\left\langle \delta\hat{\mathbf{u}} \left| \mathbf{F}(\hat{\mathbf{u}}) + 2\lambda \hat{\mathbf{u}} \right. \right\rangle = \int_{\mathcal{V}} \delta\hat{\mathbf{u}}^*(\mathbf{x}) \cdot \left[\mathbf{F}(\hat{\mathbf{u}}) + 2\lambda \hat{\mathbf{u}} \right] d^3x = 0 \quad , \quad (\text{MS.20})$$

for all possible vector-valued functions $\delta\hat{\mathbf{u}}^*(\mathbf{x})$. This means that to be an extremum $\hat{\mathbf{u}}(\mathbf{x})$ must be an eigenfunction of the operator $\mathbf{F}(\mathbf{x})$. The minimum of the ratio

$$R\{\hat{\mathbf{u}}(\mathbf{x})\} = \frac{\langle \hat{\mathbf{u}} | \mathbf{F}(\hat{\mathbf{u}}) \rangle}{\langle \hat{\mathbf{u}} | \hat{\mathbf{u}} \rangle} \quad , \quad (\text{MS.21})$$

is thus the minimum eigenvalue — the minimum value of ω^2 . This is a standard method to analyze Stürm-Liouville equations.

The foregoing provides a way to demonstrate instability without solving the full eigenvalue eq. (MS.5). If any function $\hat{\mathbf{w}}(\mathbf{x})$ can be found for which

$$W\{\hat{\mathbf{w}}\} = \frac{1}{8\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0) \right|^2 - \alpha(\mathbf{x}) (\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0) \right\} d^3x \quad , \quad (\text{MS.22})$$

is negative, then there must be at least one negative eigenvalue ω^2 . In particular the minimum eigenvalue, $\omega_0^2 < 0$. Corresponding to this will be a pair of imaginary eigenvalues

$$\omega_0 = \pm i\Omega_0 \quad , \quad \Omega_0 \geq |W\{\hat{\mathbf{w}}\}| \quad . \quad (\text{MS.23})$$

The eigenmode with the upper sign will grow exponentially in time, $e^{-i\omega_0 t} = e^{\Omega_0 t}$. This mode is unstable. Its existence is demonstrated by finding a single negative value of W . This is far easier to do than it would be to find actual eigenvalues of eq. (MS.5) and verify that one is negative.

We show below that W is the second variation (i.e. second derivative) of the total magnetic energy. It is the direct analog of the spring constant for this continuum system. A negative spring constant occurs for an equilibrium at a maximum or saddle of its potential energy functions, instead of a minimum. The same is true here: instability occurs when there is a way to perturb the system from equilibrium which *lowers* its magnetic (potential) energy.

It is far more difficult to establish *stability*, since one must demonstrate than *no function* exists capable of making W negative. Such a demonstration is, however, straight-forward for a potential field, $\nabla \times \mathbf{B} = 0$, regardless of how complicated it might be. In this case $\alpha(\mathbf{x}) = 0$ and the second term eq. (MS.22) vanishes leaving

$$W\{\hat{\mathbf{w}}\} = \frac{1}{8\pi} \int_{\mathcal{V}} \left| \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0) \right|^2 d^3x \quad . \quad (\text{MS.24})$$

Its form makes it clear that W can never be negative. This demonstrates that, at low plasma- β , *any potential magnetic field is a stable equilibrium*.

The first term in the integral eq. (MS.22) will never be negative; it is referred to as a *stabilizing* contribution. Instability therefore requires that the second term, the *destabilizing*

term, be negative and larger than the first for some choice of function $\hat{\mathbf{w}}(\mathbf{x})$. This second term is proportional to the equilibrium magnetic twist, $\alpha(\mathbf{x})$, so evidently equilibrium twist of sufficient magnitude is necessary for instability. It multiplies a factor

$$(\hat{\mathbf{w}} \times \mathbf{B}_0)^* \cdot \nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0) ,$$

which will change sign if the factor $\hat{\mathbf{w}}(\mathbf{x}) \times \mathbf{B}_0(\mathbf{x})$ undergoes a mirror reflection: this would change the sign of the curl. This suggests that the equilibrium will be unstable to perturbations of a particular *handedness* and stable to perturbations of the other handedness. The unstable handedness will depend on the sign of α , since changing that along with the mirror reflection will leave the destabilizing term unchanged.

The destabilizing term involves a single derivative of $\hat{\mathbf{w}}(\mathbf{x})$ while the stabilizing term involves two. It is for this reason that eigenfrequencies in the Eikenol limit are purely real. Unstable modes must have spatial variations comparable to, or longer than, the equilibrium field.

MS.2 Instability of a cylindrical constant- α field — the kink mode

We illustrate the stability analysis using a constant- α field in cylindrical geometry

$$\mathbf{B}_0 = B_{00} \left[J_0(\alpha r) \hat{\mathbf{z}} + J_1(\alpha r) \hat{\phi} \right] , \quad (\text{MS.25})$$

where J_m is the Bessel function. We consider this field inside a cylindrical domain bounded by a rigid conductor at $r = a$ and periodic in z with period L . This places no restriction on α but will play a role in restricting allowed perturbations. Because α is constant it is possible to evaluate W for perturbations which take the form

$$\hat{\mathbf{w}} \times \mathbf{B}_0 = \mathbf{Q}(\mathbf{x}) + \nabla \psi , \quad (\text{MS.26})$$

where $\mathbf{Q}(\mathbf{x})$ has the property

$$\nabla \times \mathbf{Q} = \mu \mathbf{Q} , \quad (\text{MS.27})$$

for some constant μ . Putting this into eq. (MS.22) gives

$$\begin{aligned} W\{\hat{\mathbf{w}}\} &= \frac{1}{8\pi} \mu(\mu - \alpha) \int_{\mathcal{V}} |\mathbf{Q}|^2 d^3x - \frac{\alpha\mu}{8\pi} \int_{\mathcal{V}} \nabla \psi^* \cdot \mathbf{Q} d^3x \\ &= \frac{1}{8\pi} \mu(\mu - \alpha) \int_{\mathcal{V}} |\mathbf{Q}|^2 d^3x - \frac{\alpha\mu}{8\pi} \oint_{\partial\mathcal{V}} \psi^* \mathbf{Q} \cdot d\mathbf{a} + \frac{\alpha\mu}{8\pi} \int_{\mathcal{V}} \psi^* \underbrace{(\nabla \cdot \mathbf{Q})}_{=0} d^3x \\ &= \frac{1}{8\pi} \mu(\mu - \alpha) \int_{\mathcal{V}} |\mathbf{Q}|^2 d^3x \end{aligned} \quad (\text{MS.28})$$

after dropping the surface integral because $Q_r = 0$ as explained below. The integral of $|\mathbf{Q}|^2$ is clearly positive, but if the coefficient $\mu(\mu - \alpha)$ is negative the equilibrium is unstable. This will happen for $0 < \mu < \alpha$, or $\alpha < \mu < 0$ depending on the sign of α . We thus see that the sign of μ , which specifies the handedness of $\nabla \times (\hat{\mathbf{w}} \times \mathbf{B}_0)$, must match that of α , as predicted above.

The value of μ is fixed by the requirement that $\mathbf{Q}(\mathbf{x})$ satisfy (MS.27) within the $a \times L$ cylinder. Taking a second curl yields the vector Helmholtz equation

$$\nabla^2 \mathbf{Q} = -\mu^2 \mathbf{Q} . \quad (\text{MS.29})$$

Because the domain is invariant under translation in z and rotation of ϕ we may propose solutions $\sim e^{ikz+im\phi}$. The periodic boundary conditions demand that

$$k = k_n = \frac{2\pi n}{L} , \quad n = 0, \pm 1, \pm 2, \dots . \quad (\text{MS.30})$$

Solutions of the vector Helmholtz equation with this symmetry, sometimes called *Chandrasekhar-Kendall functions* are

$$\mathbf{Q}(r, \phi, z) = \left[J_m \hat{\mathbf{z}} - \frac{k_n J_{m+1} + (\mu + k_n) J'_m}{\sqrt{\mu^2 - k_n^2}} \hat{\phi} \right] e^{ik_n z + im\phi} \quad (\text{MS.31})$$

$$+ i \frac{\mu J_{m+1} + (k_n + \mu) J'_m}{\sqrt{\mu^2 - k_n^2}} \hat{\mathbf{r}} \Big] e^{ik_n z + im\phi} \quad (\text{MS.32})$$

where all Bessel functions and their derivatives are evaluated at $r\sqrt{\mu^2 - k_n^2}$. The $\hat{\mathbf{z}}$ component is a solution of the scalar Helmholtz equation because the vector Laplacian is the same as the scalar Laplacian of any cartesian component; the other components are more complex.

The spectrum of μ is determined by the boundary conditions on \mathbf{Q} which is related to $\hat{\mathbf{w}}$ through eq. (MS.26). This involves $\nabla\psi$ about which we do not know. That factor is essential in order that

$$\mathbf{B}_0 \cdot (\hat{\mathbf{w}} \times \mathbf{B}_0) = \mathbf{B}_0 \cdot \mathbf{Q} + \mathbf{B}_0 \cdot \nabla\psi = 0 . \quad (\text{MS.33})$$

As a result of this condition ψ will also $\sim e^{ik_n z + im\phi}$ and

$$\nabla\psi = \frac{\partial\psi}{\partial r} \hat{\mathbf{r}} + i \left[\frac{m}{r} \hat{\phi} + k_n \hat{\mathbf{z}} \right] \psi = \frac{\partial\psi}{\partial r} \hat{\mathbf{r}} + i\boldsymbol{\kappa} \psi . \quad (\text{MS.34})$$

A notable exception occurs were we to choose $m = n = 0$ so ψ depends on r alone. According to eq. (MS.33), there is no displacement capable of producing this change unless $\mathbf{B}_0 \cdot \mathbf{Q} = 0$ everywhere, which will not occur in general. We henceforth exclude the particular case $m = n = 0$.

The radial displacement can be isolated using the components

$$(\hat{\mathbf{r}} \times \boldsymbol{\kappa}) \cdot (\hat{\mathbf{w}} \times \mathbf{B}_0) = \hat{w}_r (\boldsymbol{\kappa} \cdot \mathbf{B}_0) = (\hat{\mathbf{r}} \times \boldsymbol{\kappa}) \cdot \mathbf{Q} . \quad (\text{MS.35})$$

The outer boundary, $r = a$, is rigid and so \hat{w}_r must vanish there, meaning

$$\begin{aligned} (\hat{\mathbf{r}} \times \boldsymbol{\kappa}) \cdot \mathbf{Q} &= \frac{m}{a} J_m + \frac{k_n^2 J_{m+1} + k_n (\mu + k_n) J'_m}{\sqrt{\mu^2 - k_n^2}} = 0 \\ &= \frac{\mu}{\sqrt{\mu^2 - k_n^2}} \left[\mu J_{m+1} + (k_n + \mu) J'_m \right] \\ &= \frac{\mu}{2\sqrt{\mu^2 - k_n^2}} \left[(\mu - k_n) J_{m+1} + (k_n + \mu) J_{m-1} \right] \end{aligned} \quad (\text{MS.36})$$

where Bessel functions are evaluated at $a\sqrt{\mu^2 - k_n^2}$. It is evident from the second line that Q_r also vanishes at the outer boundary (not by coincidence) which justifies dropping the surface term in eq. (MS.28).

The outer boundary condition leads to expression (MS.36) which is a transcendental equation for μ of the form

$$S_{m,n}(\mu a) = \sqrt{\frac{\mu a - nk_1 a}{\mu a + nk_1 a}} J_{m+1} \left(\sqrt{\mu^2 a^2 - n^2 k_1^2 a^2} \right) + \sqrt{\frac{\mu a + nk_1 a}{\mu a - nk_1 a}} J_{m-1} \left(\sqrt{\mu^2 a^2 - n^2 k_1^2 a^2} \right) = 0 \quad (\text{MS.37})$$

The function $S_{m,n}$ depends only on the the aspect ratio of the cylindrical domain: $k_1 a = 2\pi a/L$. For a given aspect ratio the functions of μa can be plotted for each choice of integers m and n to find the spectrum of values $\mu_{m,n}(a/L)$. This is done in fig. 1 for the case $a/L = 1/4$.

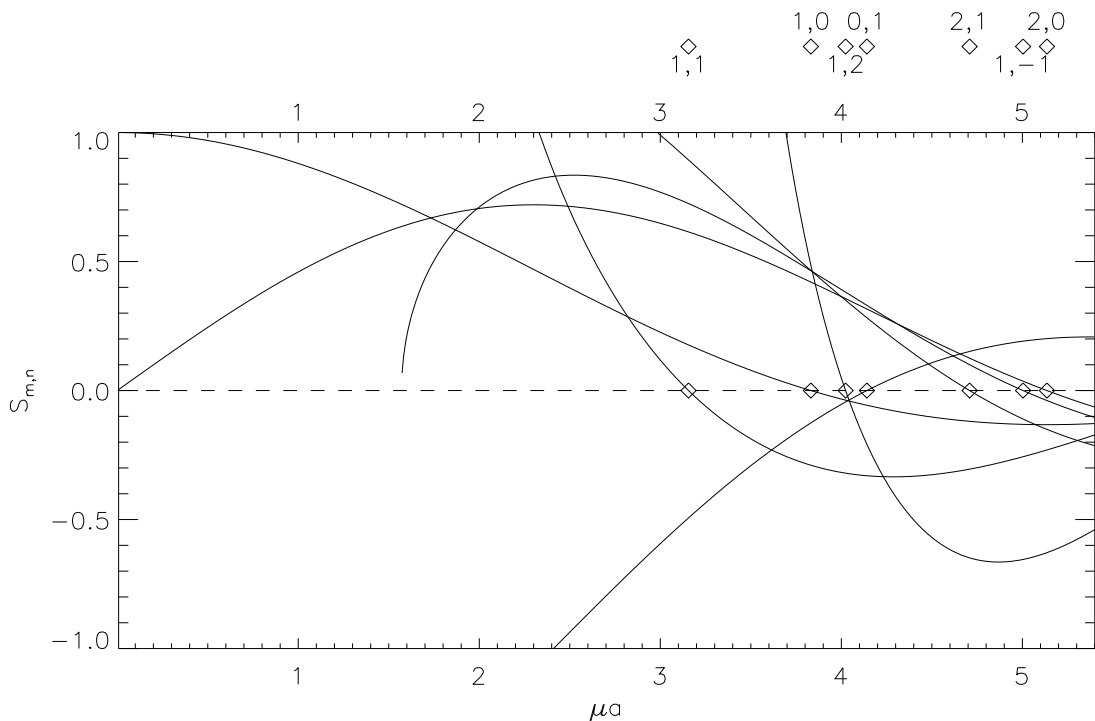


Figure 1: Plots of the curves $S_{m,n}(\mu a)$ in a cylinder with aspect ratio $a/L = 1/4$. Curves shown are for $(m, n) = (1, 1), (1, 0), (1, 2), (0, 1), (2, 1), (1, -1), (2, 0)$. Their intersection with zero is marked with a diamond and this location is reproduced, accompanied by a label, above the graphs. The left-most intersection is $\mu a = 3.156$ for $(m, n) = (1, 1)$.

Instability of the equilibrium will occur if $0 < \mu_{m,n} < \alpha$ for any pair of indices m and n . It is evident from the figure that the smallest such value occurs for $m = n = 1$ at $\mu a = 3.156$. This normal mode is called the *kink mode* due to its helical perturbation to the current channel. It turns out to be true for all aspect ratios that the kink mode has

the lowest value of μ and is therefore the mode to which the equilibrium is first unstable. Instability will occur when $\alpha > \mu_{1,1}$ meaning

$$\alpha \geq \frac{3.156}{a} \quad , \quad (\text{MS.38})$$

for the 1 : 4 aspect ratio cylinder. As predicted above, instability requires the twist α to be larger than the inverse of some characteristic length. The length is evidently the radius of the cylinder and when αa exceeds a critical value the equilibrium becomes unstable.

MS.3 Energy and stability

The total energy of a magnetic field $\mathbf{B}(\mathbf{x})$ within a region \mathcal{V} is

$$W_M\{\mathbf{B}(\mathbf{x})\} = \frac{1}{8\pi} \int_{\mathcal{V}} |\mathbf{B}(\mathbf{x})|^2 d^3x \quad . \quad (\text{MS.39})$$

To find the field with the minimum energy we demand that this energy be stationary under variation $\mathbf{B}(\mathbf{x}) \rightarrow \mathbf{B}(\mathbf{x}) + \delta\mathbf{B}(\mathbf{x})$. This variation is performed so as to preserve the normal magnetic field, $\hat{\mathbf{n}} \cdot \mathbf{B}$ at the boundary $\partial\mathcal{V}$: we require $\hat{\mathbf{n}} \cdot \delta\mathbf{B} = 0$ there. We must also perform the variation subject to the point-wise constraint $\nabla \cdot \mathbf{B} = 0$. We do this using an undetermined multiplier, $\lambda(\mathbf{x})$, which varies in space. We therefore vary

$$W_M\{\mathbf{B}(\mathbf{x})\} = \frac{1}{8\pi} \int_{\mathcal{V}} [|\mathbf{B}(\mathbf{x})|^2 + \lambda(\mathbf{x})\nabla \cdot \mathbf{B}] d^3x \quad . \quad (\text{MS.40})$$

The first variation of this is

$$\begin{aligned} \delta W_M &= \frac{1}{8\pi} \int_{\mathcal{V}} [2\mathbf{B} \cdot \delta\mathbf{B} + \lambda(\mathbf{x})\nabla \cdot \delta\mathbf{B}] d^3x \\ &= \frac{1}{8\pi} \int_{\mathcal{V}} [2\mathbf{B} - \nabla\lambda] \cdot \delta\mathbf{B} d^3x + \underbrace{\frac{1}{8\pi} \oint_{\partial\mathcal{V}} \lambda \delta\mathbf{B} \cdot d\mathbf{a}}_{=0} \quad , \quad (\text{MS.41}) \end{aligned}$$

since $\hat{\mathbf{n}} \cdot \delta\mathbf{B} = 0$ on $\partial\mathcal{V}$. The perturbation $\delta\mathbf{B}(\mathbf{x})$ is otherwise unconstrained, owing to the undetermined multiplier. Thus the only way for δW_M to vanish is for the factor in square brackets to vanish everywhere. This means

$$\mathbf{B} = 2\nabla\lambda \quad \implies \quad \nabla \times \mathbf{B} = 0 \quad . \quad (\text{MS.42})$$

This is the equation for a potential field, whose solution is unique. The extremum must therefore be a minimum. *A potential magnetic field is the field with minimum magnetic energy.* The scalar potential is related to the undetermined Lagrange multiplier which imposes the constraint $\nabla \cdot \mathbf{B} = 0$.

It is not possible to produce every possible perturbation $\delta\mathbf{B}(\mathbf{x})$ within a highly conducting plasma. This is because the field must be changed by a velocity field $\mathbf{u}(\mathbf{x})$ acting through the ideal induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \quad . \quad (\text{MS.43})$$

This equation permits changes to the magnetic field which preserve every single field line in tact — this is the frozen field line theorem which follows from eq. (MS.43). This motion can achieve a small displacement $\mathbf{x} \rightarrow \mathbf{x} + \delta\boldsymbol{\xi}(\mathbf{x})$ over a time δt with a steady flow

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\delta t} \delta\boldsymbol{\xi}(\mathbf{x}) \quad . \quad (\text{MS.44})$$

This will change the magnetic field to

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \mathbf{B}_0(\mathbf{x}) + \delta t \left. \frac{\partial \mathbf{B}}{\partial t} \right|_{t=0} + \frac{\delta t^2}{2} \left. \frac{\partial^2 \mathbf{B}}{\partial t^2} \right|_{t=0} + \dots \\ &= \mathbf{B}_0 + \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) + \frac{1}{2} \nabla \times \left[\delta\boldsymbol{\xi} \times \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right] + \dots \quad . \quad (\text{MS.45}) \end{aligned}$$

The second term, linear in $\delta\boldsymbol{\xi}$,

$$\delta\mathbf{B}(\mathbf{x}) = \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \quad , \quad (\text{MS.46})$$

is the first order the perturbation to the magnetic field created by displacement $\delta\boldsymbol{\xi}(\mathbf{x})$. It automatically satisfies the constraint $\nabla \cdot \delta\mathbf{B} = 0$, so the Lagrange multiplier is unnecessary. For any choice of the vector field, $\delta\boldsymbol{\xi}(\mathbf{x})$, this change in magnetic field will preserve all field lines. It is therefore a subset of the arbitrary perturbations used in eq. (MS.41).

The first variation of the magnetic field with this more limited perturbation is

$$\begin{aligned} \delta W_M &= \frac{1}{8\pi} \int_{\mathcal{V}} 2 \mathbf{B}_0 \cdot \delta\mathbf{B} \, d^3x = \frac{1}{4\pi} \int_{\mathcal{V}} \mathbf{B}_0 \cdot \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \, d^3x \\ &= -\frac{1}{4\pi} \oint_{\partial\mathcal{V}} \mathbf{B}_0 \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \cdot d\mathbf{a} + \frac{1}{4\pi} \int_{\mathcal{V}} (\nabla \times \mathbf{B}_0) \cdot (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \, d^3x \\ &= \frac{1}{4\pi} \int_{\mathcal{V}} \left[\mathbf{B}_0 \times (\nabla \times \mathbf{B}_0) \right] \cdot \delta\boldsymbol{\xi}(\mathbf{x}) \, d^3x \quad , \quad (\text{MS.47}) \end{aligned}$$

after discarding surface terms since $\delta\boldsymbol{\xi} = 0$ there. For this to vanish under all possible displacements, $\delta\boldsymbol{\xi}(\mathbf{x})$, it is necessary that the factor in square brackets vanish. This requires that

$$(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = 0 \quad ; \quad (\text{MS.48})$$

the magnetic field must be force free. A force-free magnetic field is an extremum of the magnetic energy under arbitrary perturbations, (MS.46), which preserve magnetic field lines: so-called *ideal perturbations*. Since there are a more limited set of changes than the arbitrary variations in eq. (MS.41), the force-free field will have higher magnetic energy than the potential field.

The force-free field is an extremum of the energy. Whether it is a minimum, maximum or saddle depends on the second variation — the expansion of W_M to second order,

$$\delta^2 W_M = \frac{1}{8\pi} \int_{\mathcal{V}} \left[\delta\mathbf{B} \cdot \delta\mathbf{B} + 2\mathbf{B}_0 \cdot \delta^2 \mathbf{B}_0 \right] d^3x \quad , \quad (\text{MS.49})$$

where $\delta^2\mathbf{B}$ is the contribution quadratic in $\delta\boldsymbol{\xi}$ in eq. (MS.45). Inserting this and $\delta\mathbf{B}$ gives

$$\begin{aligned}
\delta^2 W_M &= \frac{1}{8\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right|^2 + \mathbf{B}_0 \cdot \nabla \times \left[\delta\boldsymbol{\xi} \times \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right] \right\} d^3x \\
&= \frac{1}{8\pi} \int_{\mathcal{V}} \left| \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right|^2 d^3x - \frac{1}{8\pi} \oint_{\partial\mathcal{V}} \mathbf{B} \times \left[\delta\boldsymbol{\xi} \times \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right] \cdot d\mathbf{a} \\
&\quad + \frac{1}{8\pi} \int_{\mathcal{V}} (\nabla \times \mathbf{B}_0) \cdot \left[\delta\boldsymbol{\xi} \times \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right] d^3x \\
&= \frac{1}{8\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right|^2 - \left[\delta\boldsymbol{\xi} \times (\nabla \times \mathbf{B}_0) \right] \cdot \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right\} d^3x \\
&= \frac{1}{8\pi} \int_{\mathcal{V}} \left\{ \left| \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right|^2 - \alpha (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \cdot \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0) \right\} d^3x \quad .(\text{MS.50})
\end{aligned}$$

This is the same quadratic form as $W\{\delta\boldsymbol{\xi}\}$ in eq. (MS.22). In this case the displacement, $\delta\boldsymbol{\xi}(\mathbf{x})$, is a real function, so complex conjugation is unnecessary. If there is a displacement which makes this second variation negative then the equilibrium is not a minimum: it has at least one direction along which it is maximum.

It is for this reason that such an equilibrium is unstable. We also see now that the linear operator \mathbf{F} is self-adjoint because it arose from the second variation of a scalar potential: it is a second derivative. It is directly analogous to the second derivative of any potential energy which, if it is negative, produces an exponentially growing solution in the linearized dynamics. The exponentially growing kinetic energy is a reflection of a decrease in the magnetic energy (i.e. potential energy). The magnetic perturbation, $\mathbf{B}_1(\mathbf{x}, t)$, grows exponentially in time, but is structured so as to *decrease* the magnetic energy.