

MHD Equilibria

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Equilibria solutions of the MHD equations, (M-C)—(M-I), which are static ($\mathbf{v} = 0$) must satisfy the single equation

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} = 0 \quad , \quad (\text{FF.1})$$

where we have included gravity. In the case $\beta \gg 1$ the magnetic contribution (the Lorentz force) can be neglected, leaving the standard hydrostatic equations. We have already discussed these at length so we will consider the opposite limit, $\beta \ll 1$ where the magnetic field is dominant. Provided that the solutions are $L \ll H_p/\beta$, where H_p is the pressure scale height, we can drop the gravitational term as well as the pressure term leaving the equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \quad . \quad (\text{FF.2})$$

Equation (FF.2) is a constraint on the magnetic field alone. A field that satisfies this condition is known as a *force-free field*¹. The condition requires that $\mathbf{B}(\mathbf{x})$ be a vector field which is everywhere parallel to its own curl: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$. The constant of proportionality, α , can be a function of space in general. There is a condition related to maintaining the requirement $\nabla \cdot \mathbf{B} = 0$. *To wit*

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \nabla \cdot (\alpha \mathbf{B}) = \mathbf{B} \cdot \nabla \alpha + \underbrace{\alpha \nabla \cdot \mathbf{B}}_{=0} \quad .$$

A force-free field must therefore satisfy the set of equations

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \quad (\text{FF.3})$$

$$\mathbf{B} \cdot \nabla \alpha = 0 \quad (\text{FF.4})$$

$$\nabla \cdot \mathbf{B} = 0 \quad . \quad (\text{FF.5})$$

These are equations for the unknown fields, $\mathbf{B}(\mathbf{x})$ and $\alpha(\mathbf{x})$, and are non-linear in these. Solutions are known as *non-linear force-free fields* (NLFFs). Since they are non-linear in the unknowns they are extremely difficult to solve. Several assumptions are typically made to make them more solvable we cover these below.

FF.1 Potential fields

The most common assumption is that there is no current within the corona: $\nabla \times \mathbf{B} = 0$. When coupled with the requirement that $\nabla \cdot \mathbf{B} = 0$ the magnetic field satisfies the same conditions associated with an electrostatic field in vacuum. The same methods can thus be used to solve for the field. Because $\nabla \times \mathbf{B} = 0$ we can express the magnetic field using a scalar potential, $\chi(\mathbf{x})$,

$$\mathbf{B}(\mathbf{x}) = -\nabla \chi \quad , \quad (\text{FF.6})$$

¹All equilibria are by definition force-free. The term force-free field refers to the magnetic field which is in equilibrium by itself.

motivating the term *potential field*. In order to satisfy the second, equation the scalar potential must satisfy Laplace's equation,

$$\nabla \cdot \mathbf{B} = -\nabla^2 \chi = 0 . \quad (\text{FF.7})$$

A major attraction of potential fields is the many well-known methods of solving Laplace's equation. Given a vertical field along a plane the potential can be found using a Green's function.

$$\chi(x, y, z) = \frac{1}{2\pi} \int \frac{B_z(x', y', 0) dx' dy'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} . \quad (\text{FF.8})$$

When the vertical field vanishes outside of a small region centered at the origin, a case approximating a sunspot, the potential field away from that region

$$\mathbf{B}(\mathbf{x}) \simeq \frac{\Phi}{2\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3} , \quad \Phi = \int B_z(x, y, 0) dx dy , \quad (\text{FF.9})$$

analogous to the electric field from a charge concentration. In this sense sunspots appear to the space above the solar surface like point magnetic charges,² at least when that space is assumed to be current free.

The global magnetic field of the Sun is often approximated using a potential field, this time outside the solar surface. The influence of plasma pressure and solar wind flow cannot be neglected to all radii: these forces are, after all, responsible for opening to let out the solar wind. The simplest means of accommodating this complex effect is to designate a spherical surface, known as the *source surface*, at $r = R_{ss}$, where the field is required to be purely radial

$$B_\theta = -\frac{1}{r} \frac{\partial \chi}{\partial \theta} \Big|_{r=R_{ss}} = 0 , \quad B_\phi = -\frac{1}{r \sin \theta} \frac{\partial \chi}{\partial \phi} \Big|_{r=R_{ss}} = 0 . \quad (\text{FF.10})$$

This can be assured by satisfying the homogeneous Dirichlet condition,

$$\chi(R_{ss}, \theta, \phi) = 0 . \quad (\text{FF.11})$$

The solution to Laplace's equation can be written using spherical harmonics

$$\chi(r, \theta, \phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{m=\ell} \left(\frac{R_{ss}^{2\ell+1}}{r^{\ell+1}} - r^\ell \right) Y_\ell^m(\theta, \phi) (g_{\ell m} + ih_{\ell m}) , \quad (\text{FF.12})$$

where $g_{\ell m}$ and $h_{\ell m}$ are real coefficients. The Neumann condition at the solar surface, $r = R_\odot$,

$$B_r(R_\odot, \theta, \phi) = \sum_{\ell, m} R_\odot^{\ell-1} \left[(\ell+1) \frac{R_{ss}^{2\ell+1}}{R_\odot^{2\ell+1}} + \ell \right] Y_\ell^m(\theta, \phi) (g_{\ell m} + ih_{\ell m}) , \quad (\text{FF.13})$$

can be used to determine the coefficients given a map of the radial field at all points of the surface at one time. Such a solution is known as a *potential field source-surface* (PFSS) model.

²It is actually the end of a long tube of magnetic flux confined by high pressure beneath the solar surface. This makes it mathematically similar to a model of a magnetic monopole proposed by Dirac.

An axisymmetric potential dipole, $\ell = 1$ and $m = 0$, is generated by the flux function

$$f(r, \theta) = \frac{\Phi_{\text{open}}}{6\pi} \left(\frac{2R_{ss}}{r} + \frac{r^2}{R_{ss}^2} \right) \sin^2 \theta , \quad (\text{FF.14})$$

whose contours are shown in fig. 1. The magnetic field

$$\begin{aligned} \mathbf{B} &= \nabla f \times \nabla \phi = \frac{1}{r^2 \sin \theta} \frac{\partial f}{\partial \theta} \hat{\mathbf{r}} - \frac{1}{r \sin \theta} \frac{\partial f}{\partial r} \hat{\boldsymbol{\theta}} \\ &= \frac{\Phi_{\text{open}}}{3\pi R_{ss}^2} \left[\hat{\mathbf{r}} \left(\frac{2R_{ss}^3}{r^3} + 1 \right) \cos \theta + \hat{\boldsymbol{\theta}} \left(\frac{R_{ss}^3}{r^3} - 1 \right) \sin \theta \right] , \end{aligned}$$

clearly satisfies condition (FF.10). That it is a potential field, $\nabla \times \mathbf{B} = 0$, can be verified by taking the curl of the expression. Integrating the radial field at the source surface, $r = R_{ss}$, over either hemisphere yields the total open magnetic flux, Φ_{open} .

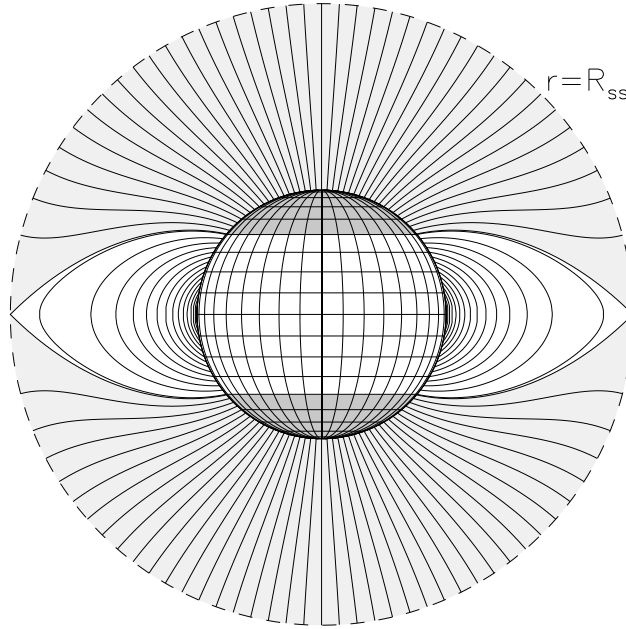


Figure 1: Field lines from the dipole PFSS field generated by eq. (FF.14), for $R_{SS} = 2.5R_{\odot}$. Field lines are initialized at polar angles spaced evenly, 2.5° apart. The shaded region contains open field lines — they connect to the source surface. The boundary between open and closed field lines, the separatrix, is anchored to polar angle $\theta_{\text{sx}} = 50^\circ$ (i.e. $\pm 40^\circ$ latitude). The regions poleward of these boundaries, the *coronal holes*, are shaded more darkly.

Field lines in a PFSS equilibrium are closed if both ends are rooted into the surface, $r = R_{\odot}$, or open if it intersects the source surface $r = R_{ss}$; the shaded region in fig. 1 consists of open field lines. The boundary between these classes of field line is a surface called a *separatrix*. In the dipole field the separatrix connects to the circle of magnetic null

points ($B_r = B_\theta = B_\phi = 0$) along the equator of the source surface ($r = R_{ss}$ and $\theta = \pi/2$). This surface is defined by the level set

$$f(r, \theta) = f_{sx} = f(R_{ss}, 0) = \frac{\Phi_{\text{open}}}{2\pi} .$$

This connects to the solar surface R_\odot at latitude θ_{sx}

$$\sin^2 \theta_{sx} = \frac{3}{2R_{ss}/R_\odot + R_\odot^2/R_{ss}^2} = \frac{3R_\odot}{2R_{ss}} \frac{1}{1 + R_\odot^3/2R_{ss}^3} . \quad (\text{FF.15})$$

The case $R_{ss} = 2.5R_\odot$ has a separatrix at $\theta_{sx} = 49.7^\circ$. The open magnetic field lines, anchored to polar angles, $\theta < \theta_{sx}$ and $\theta > \pi - \theta_{sx}$ are the source of the entire solar wind in this simple configurations. These polar caps are referred to as *coronal holes*, and shown with darker shading in fig. 1.

FF.2 Constant- α fields

The next level of sophistication is the constant- α field. As the name suggests, α is taken to be spatially uniform — it is a parameter. This choice satisfies eq. (FF.4) immediately since $\nabla\alpha = 0$. We then take a curl of eq. (FF.3) to get

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B} . \quad (\text{FF.16})$$

The Laplacian of a vector is shorthand for Laplacians of each *Cartesian* component

$$\nabla^2 B_x = -\alpha^2 B_x \quad (\text{FF.17})$$

$$\nabla^2 B_y = -\alpha^2 B_y \quad (\text{FF.18})$$

$$\nabla^2 B_z = -\alpha^2 B_z ; \quad (\text{FF.19})$$

each of these is a *Helmholtz* equation. They are linear in the unknown, so a constant- α field is also known as a *linear force-free field*.

Some feel for constant- α field comes from a solution over a periodic $L \times L$ box in the $z = 0$ plane. The vertical field at the lower boundary can be expressed as a Fourier series

$$B_z(x, y, 0) = \sum_{\mathbf{k}} \hat{b}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}_\perp} , \quad (\text{FF.20})$$

where the wave vectors

$$\mathbf{k} = \frac{2\pi}{L} m \hat{\mathbf{x}} + \frac{2\pi}{L} n \hat{\mathbf{y}} , \quad m, n \text{ integers} , \quad (\text{FF.21})$$

form a grid in the (x, y) plane and $\mathbf{x}_\perp = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is the position in the $z = 0$ plane. The coefficients of the expansion are

$$\hat{b}_{\mathbf{k}} = \frac{1}{L^2} \int_0^L \int_0^L B_z(x, y, 0) e^{-i\mathbf{k} \cdot \mathbf{x}_\perp} dx dy . \quad (\text{FF.22})$$

The most reasonable solution to eq. (FF.19) is

$$B_z(x, y, z) = \sum_{\mathbf{k}} \hat{b}_{\mathbf{k}} \exp \left[i\mathbf{k} \cdot \mathbf{x}_\perp - z \sqrt{|\mathbf{k}|^2 - \alpha^2} \right] , \quad (\text{FF.23})$$

since it will decay with height provided the radical is real: i.e. $|\alpha| < |\mathbf{k}|$. If this does not happen, because there is some wave-number whose magnitude is *smaller* than $|\alpha|$, then the magnetic field strength will not decrease with height. As a consequence the magnetic energy density $|\mathbf{B}|^2/8\pi$ will not decrease and if the box extends infinitely high the magnetic energy within it will be infinite. This is an unphysical situation.

The grid of wave numbers in eq. (FF.21), includes $\mathbf{k} = 0$ which will be smaller than any possible values of $|\alpha|$ except if $\alpha = 0$ — i.e. a potential field. The coefficient of this term

$$\hat{b}_0 = \frac{1}{L^2} \int_0^L \int_0^L B_z(x, y, 0) dx dy = \frac{\Phi}{L^2} , \quad (\text{FF.24})$$

will vanish provided the $L \times L$ box contains *no net flux*. This then is a requirement for realizable linear force free field. The next largest wave number is $|\mathbf{k}| = 2\pi/L$ from either $(m, n) = (\pm 1, 0)$ or $(m, n) = (0, \pm 1)$. It is unlikely that the coefficients of *both* of these will vanish so a linear force-free field is realistic only when the parameter falls in the range

$$|\alpha| \leq \frac{2\pi}{L} . \quad (\text{FF.25})$$

This is possible within the finite box but taking $L \rightarrow \infty$ to obtain a solution in an unbounded domain leaves only a potential field, $\alpha = 0$. While potential fields are possible in unbounded domains, linear force-free fields are possible only in bounded domains, and then only for sufficiently small twist $|\alpha|$.

For any value of α within the permitted range there is a a unique field for any flux-balanced (i.e. $\Phi = 0$) distribution of vertical field, $B_z(x, y, 0)$. The horizontal components of this field

$$\mathbf{B}_\perp(x, y, z) = \sum_{\mathbf{k}} \frac{i\hat{b}_{\mathbf{k}}}{k^2} \left(\alpha \mathbf{k} \times \hat{\mathbf{z}} - \mathbf{k} \sqrt{k^2 - \alpha^2} \right) \exp \left[i\mathbf{k} \cdot \mathbf{x}_\perp - z \sqrt{|\mathbf{k}|^2 - \alpha^2} \right] . \quad (\text{FF.26})$$

are determined entirely from the vertical field at the boundary. Even at the $z = 0$ surface the horizontal components are determined from the vertical component.

FF.3 Equilibria with symmetry: the Grad-Shafranov equation

When there is a symmetry direction with coordinate ζ , the magnetic field can be written in terms of a flux function $f(\mathbf{x})$. If the field has a component in the symmetry direction it can be written

$$\mathbf{B} = \nabla f \times \nabla \zeta + B_\zeta \nabla \zeta . \quad (\text{FF.27})$$

Field lines remain within surfaces where $f(\mathbf{x})$ is constant.

The Lorentz force from the symmetric field can be worked out in several pieces using ample vector calculus. The first term is

$$\begin{aligned} (\nabla \times \mathbf{B}) \times (\nabla f \times \nabla \zeta) &= \left[(\nabla \times \mathbf{B}) \cdot \nabla \zeta \right] \nabla f - \left[(\nabla \times \mathbf{B}) \cdot \nabla f \right] \nabla \zeta \\ &= \nabla \cdot \left[\mathbf{B} \times \nabla \zeta \right] \nabla f - \nabla \cdot \left[\mathbf{B} \times \nabla f \right] \nabla \zeta \\ &= \nabla \cdot \left[(\nabla f \times \nabla \zeta) \times \nabla \zeta \right] \nabla f - \nabla \cdot \left[(\nabla f \times \nabla \zeta) \times \nabla f \right] \nabla \zeta \end{aligned}$$

$$\begin{aligned}
& -\nabla \cdot [B_\zeta \nabla \zeta \times \nabla f] \nabla \zeta \\
= & -\nabla \cdot [|\nabla \zeta|^2 \nabla f] \nabla f - \nabla \cdot [|\nabla f|^2 \nabla \zeta] \nabla \zeta - \nabla \cdot [B_\zeta \nabla f \times \nabla \zeta] \nabla \zeta \\
= & -\nabla \cdot [|\nabla \zeta|^2 \nabla f] \nabla f - [\nabla B_\zeta \cdot (\nabla f \times \nabla \zeta)] \nabla \zeta \quad (\text{FF.28})
\end{aligned}$$

and

$$\begin{aligned}
(\nabla \times \mathbf{B}) \times B_\zeta \nabla \zeta &= B_\zeta [\nabla \times (B_\zeta \nabla \zeta)] \times \nabla \zeta = B_\zeta [\nabla B_\zeta \times \nabla \zeta] \times \nabla \zeta \\
&= -B_\zeta |\nabla \zeta|^2 \nabla B_\zeta = -\frac{1}{2} |\nabla \zeta|^2 \nabla B_\zeta^2 . \quad (\text{FF.29})
\end{aligned}$$

Using these in the equilibrium equation

$$\begin{aligned}
4\pi \nabla p &= (\nabla \times \mathbf{B}) \times \mathbf{B} \\
&= -\nabla \cdot [|\nabla \zeta|^2 \nabla f] \nabla f - [\nabla B_\zeta \cdot (\nabla f \times \nabla \zeta)] \nabla \zeta - \frac{1}{2} |\nabla \zeta|^2 \nabla B_\zeta^2 . \quad (\text{FF.30})
\end{aligned}$$

The above equation must obtain in every direction. We first take the component in the ignorable direction by contracting with $\nabla \zeta$. Due to symmetry

$$\nabla \zeta \cdot \nabla p = \nabla \zeta \cdot \nabla f = \nabla \zeta \cdot \nabla B_\zeta^2 = 0 .$$

leaving only a single term in this component of the force

$$\nabla B_\zeta \cdot (\nabla f \times \nabla \zeta) = \nabla \zeta \cdot (\nabla B_\zeta \times \nabla f) = 0 . \quad (\text{FF.31})$$

The gradients of f and B_ζ must be parallel everywhere in the solution. It can be shown that this requires B_ζ to depend on \mathbf{x} only through a dependence on $f(\mathbf{x})$. In other words

$$B_\zeta(\mathbf{x}) = B_\zeta[f(\mathbf{x})] , \quad (\text{FF.32})$$

so that

$$\nabla B_\zeta = \frac{dB_\zeta}{df} \nabla f = B'_\zeta(f) \nabla f , \quad (\text{FF.33})$$

by the chain rule. It is a peculiarity of this problem that it involves unspecified functions of the unknown function — this is the nature of the beast. Using this in eq. (FF.30)

$$4\pi \nabla p = -\nabla \cdot [|\nabla \zeta|^2 \nabla f] \nabla f - \frac{1}{2} |\nabla \zeta|^2 \frac{dB_\zeta^2}{df} \nabla f . \quad (\text{FF.34})$$

In order for the left hand side to be parallel to ∇f everywhere it is also necessary for p to depend on \mathbf{x} only through $f(\mathbf{x})$. The flux function must then satisfy the differential equation

$$\nabla \cdot [|\nabla \zeta|^2 \nabla f] = -4\pi \frac{dp}{df} - \frac{1}{2} |\nabla \zeta|^2 \frac{dB_\zeta^2}{df} . \quad (\text{FF.35})$$

This is a single equation for the flux function $f(\mathbf{x})$, known as the Grad-Shafranov equation. It is an equation of elliptic type, but involves two, as-yet-unspecified, functions of the unknown, $p(f)$ and $B_\zeta(f)$.

Cartesian symmetry

When the symmetry direction is $\hat{\mathbf{z}}$ then $\zeta = z$, $|\nabla\zeta|^2 = 1$, and eq. (FF.35) becomes

$$\nabla^2 f = -4\pi \frac{d}{df} \left[p(f) + \frac{1}{8\pi} B_z^2 \right] = -4\pi P'(f) , \quad (\text{FF.36})$$

so the two functions combine into a single function $P'(f)$. To be an equilibrium a flux function must have the property that the contours of its Laplacian coincide with its own contours. This is not a common property of functions. Solving this equation, the Grad-Shafranov equation, requires one to find a function with this unusual property.

The property is guaranteed by certain additional symmetry. For example, any cylindrical magnetic field can be an equilibrium given the appropriate distribution of axial field and pressure. Cylindrical fields are generated by flux functions $f(r)$ where $r = \sqrt{x^2 + y^2}$ is the cylindrical radius. This generates a field

$$\mathbf{B} = B_\phi(r) \hat{\phi} + B_z(r) \hat{\mathbf{z}} , \quad (\text{FF.37})$$

where $B_\phi(r) = f'(r)$. The contours of $f(r)$ will naturally be concentric circles, as will be the contours of its Laplacian. Thus there is always a function $P'(f)$, possibly multi-valued, for which eq. (FF.36) is satisfied.

To obtain this function, in the cylindrical case, we multiply eq. (FF.36) by $f'(r)$ to get

$$f' \nabla^2 f = f'(r) f''(r) + \frac{2f'^2(r)}{r} = -4\pi f'(r) P'(f) = -4\pi \frac{d}{dr} [P(f)] . \quad (\text{FF.38})$$

Integrating this gives the pressure function

$$P(r) = p + \frac{1}{8\pi} B_z^2 = P(0) - \frac{1}{8\pi} f'^2(r) - \int_0^r \frac{f'^2(r')}{2\pi r'} dr' . \quad (\text{FF.39})$$

Replacing $f'(r)$ with B_ϕ shows that the total pressure (plasma plus magnetic) decreases outward to balance inward magnetic tension

$$p + \frac{|\mathbf{B}|^2}{8\pi} = P(0) - \int_0^r \frac{B_\phi^2(r')}{2\pi r'} dr' . \quad (\text{FF.40})$$

The salient point is that for any azimuthal field $B_\phi(r)$ there is a function $P(r)$ which makes it an equilibrium. The equilibrium may be achieved with plasma pressure alone ($B_z = 0$), axial field alone ($p = 0$, in which case it is a non-linear force-free field), or any combination of the two.

Genuinely two-dimensional solutions to eq. (FF.36) can be found for different choices of the function $P'(f)$. Choosing a linear function, $P'(f) = C f$ results in a linear equation, namely Helmholtz equation,

$$\nabla^2 f = -4\pi C f = -\alpha^2 f , \quad (\text{FF.41})$$

after defining a new constant, $\alpha^2 = 4\pi C$. The solution to this equation will depend on boundary condition. Solutions with cartesian and cylindrical boundaries are

$$\begin{aligned} f(x, y) &= \cos(kx) e^{-y\sqrt{k^2 - \alpha^2}} , \quad y \geq 0 \\ f(r, \phi) &= J_m(\alpha r) \cos(m\phi) , \end{aligned}$$

where k and m are constants, and $k^2 > \alpha^2$. In the absence of plasma pressure this choice requires

$$P(f) = \frac{B_z^2}{8\pi} = \frac{\alpha^2 f^2}{8\pi} , \quad (\text{FF.42})$$

so $B_z = \alpha f$, and the solution is a constant- α force-free field.

If, however, $P'(f)$, is chosen to be a non-linear function of its argument then eq. (FF.36) will be a non-linear PDE for $f(x, y)$. There is no general method of solving such equations, so not much can be said in general about non-linear force-free fields. There are, however, many isolated examples of functions satisfying the equation. Such a solution can be identified by taking its Laplacian and verifying that it is a function of $f(x, y)$ alone. Consider, as an example, the chain of magnetic islands generated by the flux function

$$f(x, y) = \frac{B_0}{k} \ln \left[\cosh(ky) - \xi \cos(kx) \right] , \quad (\text{FF.43})$$

whose contours are shown in fig. 2; k , B_0 and ξ are parameters and $0 \leq \xi < 1$. Differentiation of this expression reveals that

$$\nabla^2 f = \frac{(1 - \xi^2)k B_0}{\left[\cosh(ky) - \xi \cos(kx) \right]^2} = (1 - \xi^2) k B_0 e^{-2kf/B_0} . \quad (\text{FF.44})$$

It is clear from the right hand expression that $\nabla^2 f$ is a function of f alone and therefore satisfies eq. (FF.36). That function is the derivative of

$$P(f) = P(0) + \frac{B_0^2}{8\pi} (1 - \xi^2) e^{-2kf/B_0} = P(0) + \frac{B_0^2}{8\pi} \frac{1 - \xi^2}{\left[\cosh(ky) - \xi \cos(kx) \right]^2} . \quad (\text{FF.45})$$

The total pressure in eq. (FF.45) could be achieved with pressure alone ($B_z = 0$) or with axial field alone ($p = 0$, this is then a *non-linear force-free field*), or by any combination of the two.

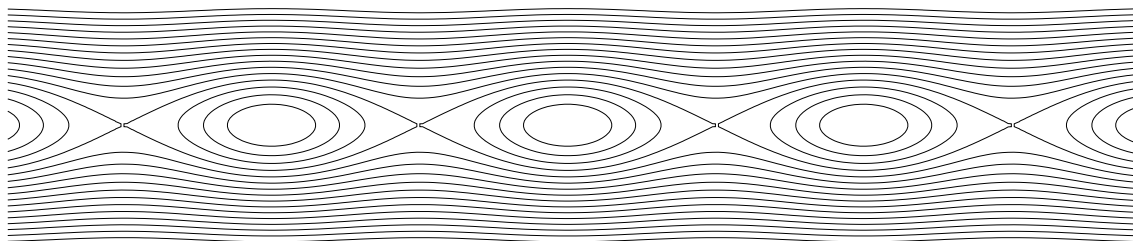


Figure 2: A chain of magnetic islands generated by the flux function eq. (FF.43).

The island-chain equilibrium is interesting by itself, but is offered here as an example of a non-linear equilibrium. It would be extraordinarily difficult to start from the non-linear PDE

$$\nabla^2 f = -e^{-f} ,$$

and obtain eq. (FF.43) as a solution. This explains the extreme scarcity of non-linear equilibria with simple expressions.