

# Shocks

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We will motivate our study of shocks by considering the simple problem of a piston moving along a cylinder of fluid initially at rest. The piston begins its motion,  $v_p \hat{\mathbf{x}}$ , at time  $t = 0$  creating a flow  $\mathbf{v} = u_x(x, t) \hat{\mathbf{x}}$  ahead of it. The fluid next to the piston will move with it,  $u_x = v_p$ , while far ahead of it the fluid will still be at rest, as yet “unaware” the piston has begun moving. There will be a moving front at which information about the oncoming piston reaches the stationary fluid, causing it to begin moving. This is the *shock front*. The natural question is: “How fast does the shock front move?” Previous experience leads to expect this information to move at the speed of sound in the unperturbed fluid,  $c_{s0}$ . But what if the piston moves even faster than that?

## SH.1 Weak shocks: sound waves from a slow piston

Let us begin with the case we already understand: piston motion far below than the sound speed,  $v_p \ll c_{s0}$ . This is the case we can legitimately treat as a small perturbation to a static homogeneous equilibrium, and solve by linearization. To accommodate conventional shock notation in the future we denote the perturbation with a  $\delta$  rather than a subscript,

$$p(x, t) = p_0 + \delta p(x, t) + \dots ,$$

and so forth. In this notation the two linearized equations which couple are

$$\partial_t \delta u_x = -\frac{1}{\rho_0} \partial_x \delta p \quad (\text{SH.1})$$

$$\partial_t \delta p = -\gamma p_0 \partial_x \delta u_x . \quad (\text{SH.2})$$

This is a set of telegraphers equations. The fluid is initially equilibrium so

$$\delta u_x(x, 0) = 0 \quad , \quad \delta p(x, 0) = 0 .$$

The perturbation is caused by the piston moving to the right at speed  $v_p$  beginning at time  $t = 0$ . This introduces a boundary condition at position  $x_p = v_p t$

$$\delta u_x(x_p, t) = v_p \Theta(t) = \begin{cases} 0 & , \quad t < 0 \\ v_p & , \quad t > 0 \end{cases} \quad (\text{SH.3})$$

where  $\Theta(t)$  is the Heaviside function. The solutions to (SH.1) and (SH.2) subject to these conditions, are

$$\delta u_x(x, t) = v_p \Theta(c_{s0}t - x) \quad (\text{SH.4})$$

$$\delta p(x, t) = p_0 \frac{\gamma v_p}{c_{s0}} \Theta(c_{s0}t - x) , \quad (\text{SH.5})$$

within the region  $x > 0$ .

By a time  $t > 0$  the piston has moved to position  $x_p = v_p t$  and the fluid near it is moving at its same speed  $v_p \ll c_{s0}$ . This co-moving fluid composes a column extending to a shock

front at  $x_s = c_{s0}t$ . The shock front moves ahead at the speed of sound,  $c_{s0}$ , much faster than the fluid in the column. The total length of the column

$$L_c = x_s - x_p = (c_{s0} - v_p)t , \quad (\text{SH.6})$$

thus increases in time. The mass density within the column

$$\begin{aligned} \rho_c &= \rho_0 + \delta\rho + \dots \simeq \rho_0 \left(1 + \frac{\delta\rho}{\rho_0}\right) \\ &= \rho_0 \left(1 + \frac{1}{\gamma} \frac{\delta p}{p_0}\right) = \rho_0 \left(1 + \frac{v_p}{c_{s0}}\right) , \end{aligned} \quad (\text{SH.7})$$

has been enhanced very slightly over its initial value — the smallness of the perturbation is required to justify our linear treatment. The mass within the enhanced column

$$\rho_c L_c = \rho_0 \left(1 + \frac{v_p}{c_{s0}}\right) (c_{s0} - v_p)t = \rho_0 c_{s0} t + \mathcal{O}\left(\frac{v_p^2}{c_{s0}^2}\right) \simeq \rho_0 x_s , \quad (\text{SH.8})$$

is the same as the mass of undisturbed fluid before the piston moved.

### Analysis in a co-moving frame

We now take a new view of our solution by transforming to a reference frame moving rightward at the speed of the shock,  $v_s = c_{s0}$ . In this reference frame the undisturbed, or *pre-shock*, fluid appears to be moving leftward at

$$u_1 = 0 - v_s = -c_{s0} , \quad (\text{SH.9})$$

(see fig. 1). The flow slows down at the shock to

$$u_2 = \delta u_x - v_s = -c_{s0} \left(1 - \frac{v_p}{c_{s0}}\right) . \quad (\text{SH.10})$$

The flow is slowed down by the enhanced pressure on the downstream side,  $p_2 > p_1$ , which goes along with the enhanced density.<sup>1</sup> The mass flux into the shock, from the right, is

$$\Gamma_\rho = \rho_1 u_1 = -\rho_0 c_{s0} . \quad (\text{SH.11})$$

The flow velocity downstream of the shock,  $u_2$ , is slower but the mass density  $\rho_2 = \rho_c$  is larger. The total flux

$$\Gamma_\rho = \rho_2 u_2 = -\rho_0 \left(1 + \frac{v_p}{c_{s0}}\right) (c_{s0} - v_p) \simeq -\rho_0 c_{s0} , \quad (\text{SH.12})$$

matches that from upstream. This is not a surprising result since mass must be conserved as it crosses the shock. It does, however, illustrate how reduced speed must be accompanied by enhanced density in this co-moving view.

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<sup>1</sup>This appears to raise a chicken-and-egg issue. The initial problem of a piston resolves this nicely.

## SH.2 Shocks: response to a fast piston

The linear treatment above required the piston to move much slower than the sound speed. Information about the approaching piston traveled at the sound speed, compressing and accelerating the fluid in advance. What would happen if the piston actually moved *faster* than sound:  $v_p > c_{s0}$ ? How would the fluid ahead of it learn it was coming? How much would the fluid be compressed by this supersonic piston?

The answer must clearly be that the piston will compress the fluid by more than a small enhancement. This will require a solution of the full nonlinear equations. The previous linear treatment gives us insight into what the solution will look like (see fig. 1). Enhancement to the density and pressure will occur at a shock front moving at some speed,  $v_s$ . Ahead of this will be the uniform, undisturbed fluid — the pre-shock region we call region 1. Behind the shock front will be a second uniform region, called region 2, with density,  $\rho_2$ , pressure  $p_2$  and velocity  $v_p$  to match the piston. Between these uniform regions will be some kind of transition between upstream and downstream states.

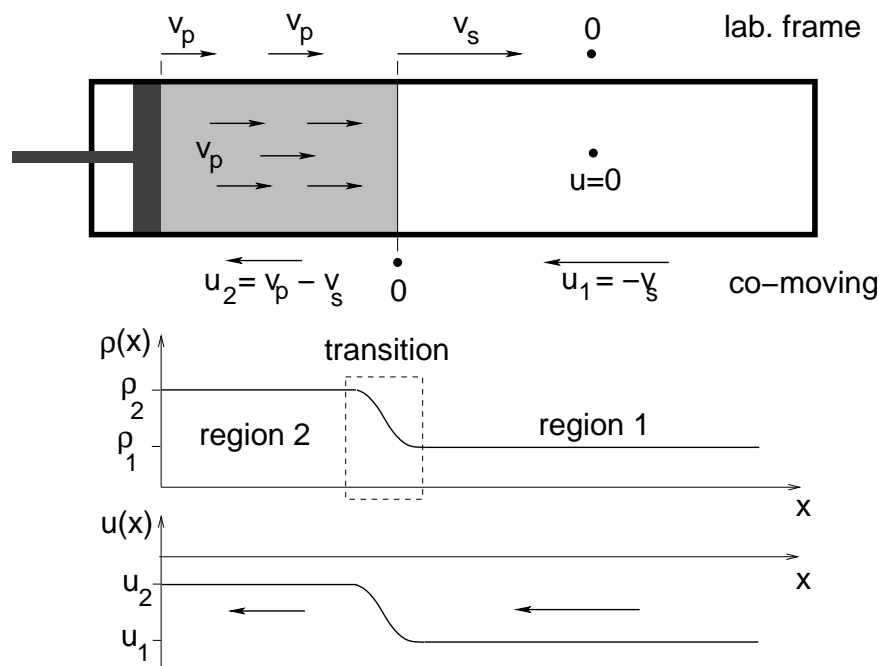


Figure 1: Geometry of the piston and shock problems. The piston is shown on top with the column of compressed gas in grey. Arrows above and below the piston show velocities in the lab reference frame (above) and in the frame co-moving with the shock front (below). Axes below plot the solutions  $\rho(x)$  and  $u(x)$  to the non-linear shock problem posed in the co-moving frame. The solution is composed uniform regions, 1 and 2, separated by a transition region (dashed box).

The classic solution to the problem is found in a reference frame moving with this shock, at  $v_s$ . In this co-moving reference frame we expect the solution to be steady:  $\partial/\partial t = 0$  for all quantities. There will be inflow from the right at  $u_1 = -v_s < 0$ ; this is how the stationary fluid appears in the co-moving frame. This inflow will have the properties of the

stationary fluid,  $\rho_1$  and  $p_1$ . Downstream of the shock will be a second uniform region with density  $\rho_2 > \rho_1$ , pressure  $p_2 > p_1$  and velocity  $u_2$  which has smaller magnitude than  $u_1$  but still negative. Moving with the piston the flow will have velocity  $u_2 = v_p - v_s < 0$ . Thus the piston problem has boundary conditions  $\rho_1, p_1$ , from the stationary fluid, and  $u_2 - u_1 = v_p$  from the piston. It is easier to consider the problem with all upstream values,  $\rho_1, u_1$ , and  $p_1$ , fixed and to solve for the downstream values,  $\rho_2, u_2$  and  $p_2$ . We will follow this approach and return in the end to show how  $v_p = u_2 - u_1$  can be used to find the correct solution for a given piston.

### Equations for the shock

We solve the problem using the standard fluid equations, assuming a steady solution,  $\partial_t = 0$ . This means that whatever shape the transition takes, it appears steady in the co-moving frame. The continuity and momentum equations in conservative form, (C) and (M-c) become

$$\partial_t \rho = -\partial_x(\rho u) = 0 \quad (\text{SH.13})$$

$$\partial_t(\rho u) = -\partial_x(\rho u^2) - \partial_x p + \partial_x\left(\frac{4}{3}\mu\partial_x u\right) = 0 \quad (\text{SH.14})$$

Viscosity will turn out to be important to a self-consistent treatment and we have used the most accurate form, eqs. (1.31) and (1.32), restricted to a single direction

$$f_x^{(\text{visc})} = \partial_x \sigma_{xx} = \partial_x \left[ \mu \left( \partial_x u + \partial_x u - \frac{2}{3} \nabla \cdot \mathbf{u} \right) \right] = \partial_x \left( \frac{4}{3} \mu \partial_x u \right) \quad (\text{SH.15})$$

The two equations thus dictate that the fluxes of mass and momentum are constant in the co-moving frame

$$\Gamma_\rho = \rho u \quad (\text{SH.16})$$

$$\Gamma_p = \rho u^2 + p - \frac{4}{3} \mu \partial_x u \quad (\text{SH.17})$$

A third relation must come from the conservation of energy. We use the conservative form of thermal energy conservation, eq. (E-c),

$$\partial_t \left( \frac{p}{\gamma - 1} \right) = -\partial_x \left( u \frac{p}{\gamma - 1} \right) - p \partial_x u + \frac{4}{3} \mu (\partial_x u)^2 = 0 \quad (\text{SH.18})$$

including viscous heating from eq. (1.42),<sup>2</sup> to recover the kinetic energy sapped by viscosity; we omit thermal conduction for simplicity. The right hand side is not a simple  $x$  derivative of a conserved flux, so it is not the kind of conservation law we need. This is because thermal energy alone is not conserved. To obtain a conservation law for *total* energy we must add the evolution of kinetic energy, found from eq. (SH.14),

$$\partial_t \left( \frac{1}{2} \rho u^2 \right) = -\partial_x \left( \frac{1}{2} \rho u^3 \right) - u \partial_x p + u \partial_x \left( \frac{4}{3} \mu \partial_x u \right) = 0 \quad (\text{SH.19})$$

Adding this to eq. (SH.18) yields an evolution for total energy density

$$\partial_t \left( \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \right) = -\partial_x \left( \frac{\gamma}{\gamma - 1} u p \right) - \partial_x \left( \frac{1}{2} \rho u^3 \right) + \partial_x \left( \frac{4}{3} \mu u \partial_x u \right) = 0 \quad (\text{SH.20})$$

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<sup>2</sup>The factor  $\frac{4}{3}$  is not trivial to arrive at. It is related to the matrix norm  $\|2\hat{\mathbf{x}}\hat{\mathbf{x}} - \frac{2}{3}\mathbf{I}\|^2 = \frac{8}{3}$ .

The right hand side of this is a perfect derivative of the energy flux which is conserved

$$\Gamma_e = \frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma-1}up - \frac{4}{3}\mu u \partial_x u . \quad (\text{SH.21})$$

The second term is recognizable as enthalpy flux, the first is a flux of kinetic energy and the third is an energy flux due to work by viscous stress.

### Rankine-Hugoniot relations

The stationary solution we seek consists of a thin transition region over which upstream quantities  $\rho_1$ ,  $u_1$  and  $p_1$  change to downstream quantities  $\rho_2$ ,  $u_2$  and  $p_2$ . Outside the region all quantities are uniform and the derivative term  $\partial_x u$  will vanish. That means that the upstream and downstream values must each be consistent with the conserved fluxes

$$\Gamma_\rho = \rho_1 u_1 = \rho_2 u_2 \quad (\text{SH.22})$$

$$\Gamma_p = \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \quad (\text{SH.23})$$

$$\Gamma_e = \frac{1}{2}\rho_1 u_1^3 + \frac{\gamma}{\gamma-1}u_1 p_1 = \frac{1}{2}\rho_2 u_2^3 + \frac{\gamma}{\gamma-1}u_2 p_2 . \quad (\text{SH.24})$$

These three conservation laws, conservation of mass, momentum and energy between upstream and downstream flows, are known as the *Rankine-Hugoniot* relations. They can be used to derive downstream quantities  $\rho_2$ ,  $u_2$  and  $p_2$  from a given set of upstream values,  $\rho_1$ ,  $u_1$  and  $p_1$ .

To facilitate this derivation we combine the fluxes into a single non-dimensional constant

$$\frac{\Gamma_p^2}{2\Gamma_e\Gamma_\rho} = \frac{(\rho u^2 + p)^2}{\rho u[\rho u^3 + 2\gamma p u/(\gamma-1)]} , \quad (\text{SH.25})$$

which must be the same on the two sides. We replace  $u = M c_s$ , where  $M$  is the local Mach number and  $c_s$  is the local sound speed, and rework the expression into a form

$$\frac{\Gamma_p^2}{2\Gamma_e\Gamma_\rho} = \frac{(\gamma-1)(\gamma M^2 + 1)^2}{\gamma^2 M^2[(\gamma-1)M^2 + 2]} = F(M) , \quad (\text{SH.26})$$

involving the local Mach number alone. The Mach number,  $M_1^2 = \rho_1 u_1^2/\gamma p_1$ , is the only dimensionless combination of state variables, so it is naturally the only one that appears in the dimensionless ratio of fluxes. The Mach number of the upstream flow is thus the single parameter characterizing the shock; from it all other shock properties follow. It is common to characterize the shock by Mach number  $M_1$ , and to call this the *Mach number of the shock*.

Since eq. (SH.26) must be the same on the two sides we can invert the function  $F(M)$  to find the downstream Mach number

$$M_2 = F^{-1}\left[F(M_1)\right] , \quad (\text{SH.27})$$

strictly in term of the upstream Mach number. The remaining downstream quantities can then be found from upstream quantities and  $M_2$ .

The plot in fig. 2 reveals that  $F^{-1}$  has two branches separated by the minimum of  $F(M)$  at  $M = 1$ . Choosing the same branch for  $F^{-1}$  as for  $F(M_1)$  would yield the trivial result of no change across the shock:  $M_2 = M_1$ . There would be no change in any quantity,  $\rho_2 = \rho_1$ ,  $u_2 = u_1$  and  $p_2 = p_1$ , which is clearly consistent with the conservation laws but is not a shock — it is not a transition. The alternative is for  $M_1$  and  $M_2$  to be on opposite sides of the minimum,  $M = 1$ , as shown in the figure. Thus a genuine shock transition must connect *supersonic* flow to *subsonic* flow. We will see below that the upstream side must be the supersonic flow,  $M_1 > 1$ , and the downstream side must be subsonic,  $M_2 < 1$ .

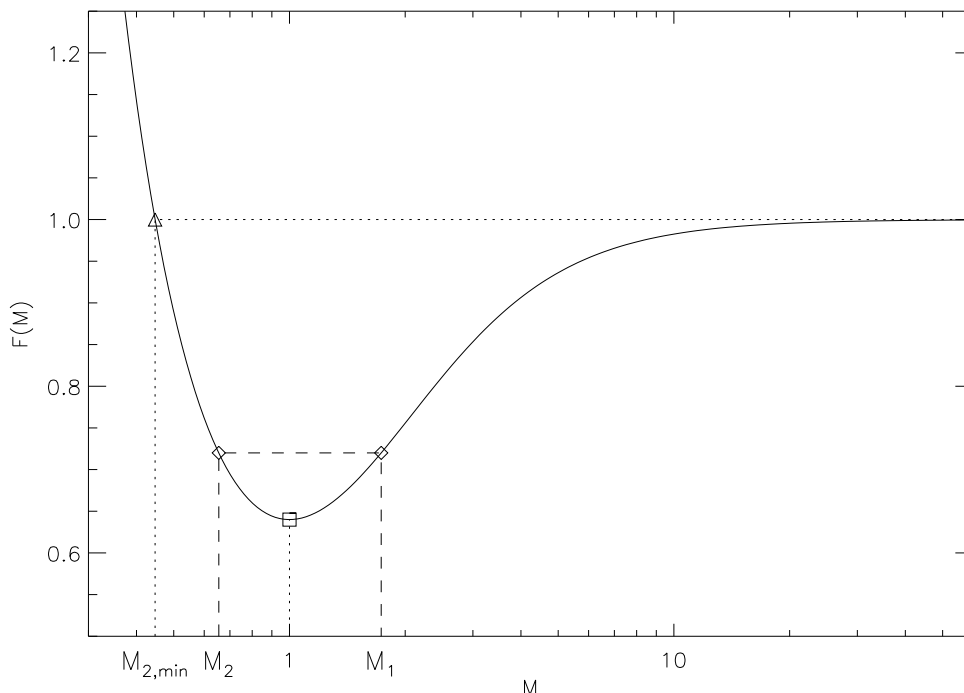


Figure 2: A plot of the function  $F(M)$  from eq. (SH.26) for  $\gamma = 5/3$ . A square marks to minimum at  $M = 1$ . Diamonds show an example of values of  $M_2$  and  $M_1$  satisfying the relation  $F(M_2) = F(M_1)$ . A horizontal dotted line shows the asymptote,  $F(M) \rightarrow 1$  as  $M \rightarrow \infty$ . This intersects the left branch of the curve at a point  $M_{2,\min} = 1/\sqrt{5}$  (triangle), from eq. (SH.36), so  $M_2$  will be restricted to the range  $M_{2,\min} < M_2 \leq 1$ .

The inversion can be performed analytically by equating the value of the functions on opposite sides

$$F(M_1) = \frac{(\gamma - 1)(\gamma M_1^2 + 1)^2}{\gamma^2 M_1^2 [(\gamma - 1)M_1^2 + 2]} = \frac{(\gamma - 1)(\gamma M_2^2 + 1)^2}{\gamma^2 M_2^2 [(\gamma - 1)M_2^2 + 2]} = F(M_2) . \quad (\text{SH.28})$$

Cross multiplying by denominators yields

$$\begin{aligned} & 2\gamma(\gamma - 1)M_2^4 M_1^2 + (\gamma - 1)M_2^4 + 2\gamma^2 M_1^4 M_2^2 + 2M_2^2 \\ &= 2\gamma(\gamma - 1)M_1^4 M_2^2 + (\gamma - 1)M_1^4 + 2\gamma^2 M_2^4 M_1^2 + 2M_1^2 , \end{aligned} \quad (\text{SH.29})$$

which is a polynomial that factors into

$$\begin{aligned} -2\gamma[M_2^4 M_1^4 - M_1^4 M_2^2] + (\gamma - 1)(M_2^4 - M_1^4) + 2(M_2^2 - M_1^2) = \\ (M_2^2 - M_1^2) [-2\gamma M_2^2 M_1^2 + (\gamma - 1)(M_2^2 + M_1^2) + 2] = 0 . \end{aligned} \quad (\text{SH.30})$$

The leading factor returns the trivial “non-transition”,  $M_1^2 = M_2^2$ . The second leads to the relation

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1} , \quad (\text{SH.31})$$

explicitly relating the downstream Mach number to the upstream Mach number. In agreement with the discussion above we see that in the limit  $M_1^2 \rightarrow 1$  the down stream Mach number  $M_2^2 \rightarrow 1$  as well — the two branches meet at the minimum. This is the limit of a weak shock, equivalent to sound wave. We treated the case in the first section finding that in a co-moving frame  $u_1 \simeq u_2 \simeq c_{s0}$ .

The remainder of the downstream properties can be derived using eq. (SH.31) in conjunction with the the original conservation laws. Equation (SH.23) dictates that

$$1 = \frac{\rho_1 u_1^2 + p_1}{\rho_2 u_2^2 + p_2} = \frac{p_1(\gamma M_1^2 + 1)}{p_2(\gamma M_2^2 + 1)} . \quad (\text{SH.32})$$

This provides an explicit expression from the pressure ratio

$$\frac{p_2}{p_1} = \frac{\gamma M_1^2 + 1}{\gamma M_2^2 + 1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} \quad (\text{SH.33})$$

where the second expression comes from using (SH.31) to replace  $M_2^2$ . We can then use the ratio

$$\frac{M_1^2}{M_2^2} = \frac{u_1^2(\rho_1/p_1)}{u_2^2(\rho_2/p_2)} = \frac{\rho_1 u_1}{\rho_2 u_2} \times \frac{u_1 p_2}{u_2 p_1} = \frac{u_1 p_2}{u_2 p_1} . \quad (\text{SH.34})$$

This leads to the ratio

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{M_1^2 p_1}{M_2^2 p_2} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} , \quad (\text{SH.35})$$

after using eqs. (SH.31) and (SH.33). These ratios are plotted in fig. 3.

Expressions (SH.31), (SH.33) and (SH.35) have noteworthy behavior in the limit of very strong shocks,  $M_1 \rightarrow \infty$ . These are

$$M_2 \rightarrow \sqrt{(\gamma - 1)/2\gamma} = M_{2,\min} \quad (\text{SH.36})$$

$$\frac{p_2}{p_1} \rightarrow \frac{2\gamma}{\gamma + 1} M_1^2 \quad (\text{SH.37})$$

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma + 1}{\gamma - 1} . \quad (\text{SH.38})$$

Least surprising is that the post-shock pressure can be arbitrarily large as the Mach number is taken to be large. The post-shock density, on the other hand, cannot exceed a fixed multiple of the initial density. For a plasma ( $\gamma = 5/3$ ) the maximum density increase is a factor of four. This also means the flow cannot slow down by more a fixed fraction of its initial value, given by the reciprocal of the density enhancement. Finally, eq. (SH.36) gives the value of  $M_{2,\min}$ , previously noted in fig. 2; for the case  $\gamma = 5/3$ ,  $M_{2,\min} = 1/\sqrt{5}$ .

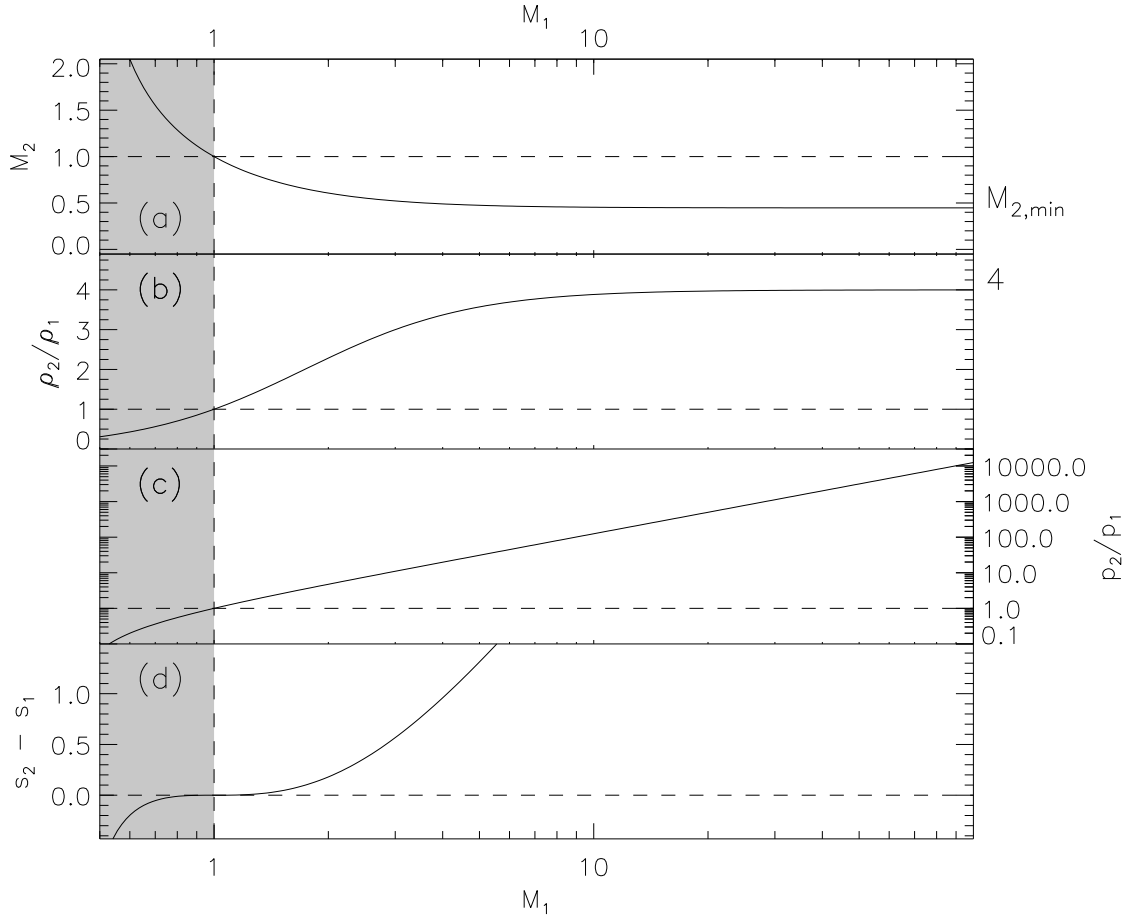


Figure 3: Plots of downstream shock properties as a function of upstream Mach number  $M_1$  for  $\gamma = 5/3$ . (a) The downstream Mach number  $M_2$  given by eq. (SH.31), (b) the density ratio  $\rho_2/\rho_1$  given by eq. (SH.35), (c) the pressure ratio  $p_2/p_1$  given by eq. (SH.33) on a logarithmic axis, and (d) the difference in entropy,  $s_2 - s_1$  given by (SH.39). The vertical dashed line shows  $M_1 = 1$ , and to its left is the unphysical region with  $M_2 > M_1$  over which the entropy change is negative.

## The entropy

The specific entropy changes as the shock is crossed

$$\begin{aligned} \frac{(\gamma - 1)}{k_B/m}(s_2 - s_1) &= \ln\left(\frac{p_2}{p_1}\right) - \gamma \ln\left(\frac{\rho_2}{\rho_1}\right) \\ &= \ln\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right] + \gamma \ln\left[1 + \frac{\gamma - 1}{\gamma + 1}(M_1^2 - 1)\right] - \gamma \ln(M_1^2) \quad , \quad (\text{SH.39}) \end{aligned}$$

plotted in fig. 3d. The difference is an increasing function of  $M_1$  which crosses  $s_2 - s_1 = 0$  at  $M_1 = 1$ . At this point the curve is very flat,

$$\frac{s_2 - s_1}{k_B/m} \simeq \frac{2\gamma}{3(\gamma + 1)^2} (M_1^2 - 1)^3 - \frac{2\gamma^2}{(\gamma + 1)^3} (M_1^2 - 1)^4 + \dots \quad ,$$

but is negative for any value of  $M_1 < 1$ .

This shows that except in the infinitesimal neighborhood of  $M_1 = 1$  the simultaneous conservation of mass, momentum and energy (i.e. eqs. [SH.22]—[SH.24]) demand there be some heating:  $\dot{Q} \neq 0$ . While those equations make no *explicit* reference to any form of heating, they cannot be solved without changing the entropy, which requires that some heating be present. Curiously the form of heating has no effect on the final result, although it must be consistent with energy conservation. For example, viscous heating enters the energy eq. (SH.20) as a perfect derivative, so it contributes to the energy flux  $\Gamma_e$  but does not add or remove energy. Viscosity converts kinetic energy into thermal energy, but the total energy of the fluid is conserved by this conversion. Thermal conduction would work similarly, adding another term to the conserved energy flux  $\Gamma_e$ . Radiation, on the other hand, is a loss term so its inclusion would make an energy conservation law like (SH.21) impossible. Radiative shocks must be analyzed using mass and momentum conservation alone.

The absence of viscosity from the results, but necessity in their derivation, raises the question of what would happen if the fast piston experiment were posed for a perfectly ideal system,  $\mu = 0$ , and  $\dot{Q} = 0$ . Clearly such a system would conserve mass, momentum and energy so eqs. (SH.22)—(SH.24) would still obtain. These constitute three equations for three unknowns,  $\rho_2$ ,  $u_2$ , and  $p_2$ , so adding a fourth constraint,

$$\frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma} \quad ,$$

would over-determine the system making solution impossible in general. The resolution of the paradox is that the initial assumption, that a solution exists which is stationary ( $\partial/\partial t = 0$ ) in a co-moving frame, becomes untenable. Instead the region downstream of the disturbance front will fail to settle into a steady, uniform structure — it will include time variations in all variables, even when viewed in a co-moving frame. Dissipative terms, such as viscosity, which convert kinetic energy to heat, turn out to be important for relaxing this motion.

The behavior of the entropy difference,  $s_2 - s_1$ , plotted in fig. 3d, provides the crucial evidence that fluid must traverse a shock from *supersonic to subsonic*:  $M_1 > M_2$ . While the alternative scenario conserves mass, momentum and energy (these laws are time-reversible),

it would require a net *decrease* in the entropy of the system, in violation of the second law of thermodynamics. It is possible for specific entropy to decrease locally, leading to a profile  $s(x)$ , with positive and negative slope.<sup>3</sup> This steady profile does not change in time so the change in the entropy of the system occurs because the mass in region 2, with specific entropy  $s_2$ , is increasing while the mass in region 1, with specific entropy  $s_1$ , is decreasing at the same rate. The rate of this transfer is given by the mass flux  $|\Gamma_\rho|$  across a shock of area  $A$ . This will change the entropy of the system at a rate

$$\frac{dS}{dt} = A|\Gamma_\rho|(s_2 - s_1) \quad , \quad (\text{SH.40})$$

which can never be negative according to the second law of thermodynamics. Thus we conclude that the solution where flow speeds up as it crosses the shock, the solutions where the fluid experiences a pressure decrease, are unphysical. Such pressure decreases do occur and are called *rarefaction waves* but they do not satisfy the assumptions of a shock: a transition which appears stationary in a co-moving frame.

### Return to the piston

The derivation above followed the classic approach in a reference frame co-moving with the shock front. It was motivated by the problem of a piston moving at speed  $v_p$  into a stationary medium with sound speed  $c_{s0}$ . This drives a shock ahead at speed  $v_s$ . The upstream medium, moving at velocity  $u_1 < 0$ , was actually the stationary medium viewed from the reference frame of the shock. We therefore have  $v_s = -u_1$ . The downstream medium, moving at  $u_2$  in the shock frame, is actually moving with the piston at

$$v_p = v_s + u_2 = u_2 - u_1 = |u_1 - u_2| \quad . \quad (\text{SH.41})$$

We can divide the piston's speed by the sound speed of the ambient medium to derive a piston Mach number<sup>4</sup>

$$M_p = \frac{v_p}{c_{s0}} = \frac{|u_1 - u_2|}{c_{s1}} = \left(1 - \frac{u_2}{u_1}\right) |M_1| = \frac{2}{(\gamma + 1)} \left(|M_1| - \frac{1}{|M_1|}\right) \quad . \quad (\text{SH.42})$$

This expression may be inverted to give the Mach number of the shock from that of the piston

$$|M_1| = M_s = \frac{1}{4}(\gamma + 1)M_p + \sqrt{\frac{1}{16}(\gamma + 1)^2 M_p^2 + 1} \quad . \quad (\text{SH.43})$$

It is evident from expression (SH.43) that a supersonic shock,  $M_s > 1$ , is created by any piston motion, no matter how slow. This contrasts with our expectation from linear theory that information will propagate at  $c_{s0}$  and never faster. The fluid is compressed by the shock, which drives up its temperature and thus its sound speed. The shock front propagates at a speed which is at once *slower* than the sound speed in the downstream medium ( $M_2 < 1$ ) and *faster* than the sound speed in the upstream (ambient) medium.

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<sup>3</sup>Such an entropy decrease does occur inside shocks when thermal conduction is included.

<sup>4</sup>In previous expression,  $M_1$  and  $M_2$ , velocities were divided by their local sound speed. Here we divide the speed of the piston by the sound speed in the *ambient* medium rather than the medium adjacent to the piston.

Extremely subsonic motion,  $M_p \ll 1$ , generates a weak shock traveling near the sound speed

$$M_s = \frac{v_s}{c_{s0}} \simeq 1 + \frac{1}{4}(\gamma + 1)M_p + \dots , \quad (\text{SH.44})$$

as we found in the first section. Placing this into expression (SH.33) yields a post-shock pressure

$$\frac{p_2}{p_1} \simeq 1 + \gamma M_p = 1 + \frac{\gamma v_p}{c_{s0}} , \quad v_p \ll c_{s0} , \quad (\text{SH.45})$$

matching eq. (SH.5) from the linear treatment.

An extremely supersonic piston will generate a shock moving at

$$M_s \simeq \frac{1}{2}(\gamma + 1)M_p , \quad v_p \gg c_{s0} . \quad (\text{SH.46})$$

The shock, in this limit, moves faster than the piston by a fixed factor

$$v_s = \frac{\gamma + 1}{2}v_p > v_p , \quad (\text{SH.47})$$

which is 4/3 for a plasma ( $\gamma = 5/3$ ). This excess speed is needed accommodate a column of compressed fluid growing at a rate

$$\dot{L}_c = v_s - v_p = \frac{\gamma - 1}{2}v_p = \frac{\gamma - 1}{\gamma + 1}v_s , \quad v_p \gg c_{s0} . \quad (\text{SH.48})$$

The front is adding new mass to the growing column with density  $\rho_1$ . It adds mass at speed  $v_s$ , so the ratio  $\dot{L}_c/v_s$  is the inverse of the limiting compression ratio  $\rho_2/\rho_1 = (\gamma + 1)/(\gamma - 1)$ , from eq. (SH.38), as required by mass conservation.

The piston head, with area  $A$ , moves at speed  $v_p$  against a force  $p_2A$ . It thus does work on the fluid at a rate  $\dot{E} = v_p p_2 A$ . The column of fluid on which it works grows at a rate  $\dot{M} = \rho_1 v_s A$ . This endows the material with a specific energy

$$\varepsilon = \frac{\dot{E}}{\dot{M}} = \frac{v_p p_2}{v_s \rho_1} = \frac{v_p}{v_s} \frac{\rho_2 p_2}{\rho_1 \rho_2} \simeq \frac{2}{\gamma + 1} \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{p_2}{\rho_2} = \frac{2}{\gamma - 1} \frac{p_2}{\rho_2} . \quad (\text{SH.49})$$

Half of this work is converted into thermal energy (per unit mass) of shocked material

$$\varepsilon_T = \frac{1}{\gamma - 1} \frac{p_2}{\rho_2} = \frac{1}{2} \varepsilon . \quad (\text{SH.50})$$

The other half is converted into bulk kinetic energy, per unit mass,

$$\begin{aligned} \frac{1}{2}v_p^2 &= \frac{1}{2}M_p^2 \gamma \frac{p_1}{\rho_1} = \frac{\gamma}{2} M_p^2 \frac{p_1}{p_2} \frac{\rho_2}{\rho_1} \frac{p_2}{\rho_2} \\ &\simeq \frac{\gamma}{2} \left( \frac{2}{\gamma + 1} |M_1| \right)^2 \left( \frac{\gamma + 1}{2\gamma M_1^2} \right) \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{p_2}{\rho_2} = \frac{1}{\gamma - 1} \frac{p_2}{\rho_2} = \frac{1}{2} \varepsilon , \end{aligned} \quad (\text{SH.51})$$

needed for the fluid to move with the piston. Remarkably, the work done by the piston is shared equally between bulk kinetic energy and internal thermal energy of the fluid.

### SH.3 Inner structure of the shock

The shock is not a discontinuous jump between states, but a transition whose shape can be found using the constancy of the fluxes including the viscous stress and heating. Equation (SH.17) can be used to replace local pressure profile

$$p(x) = \Gamma_p - \Gamma_\rho u + \frac{4}{3}\mu\partial_x u . \quad (\text{SH.52})$$

Using this in eq. (SH.21) gives

$$\begin{aligned} (\gamma - 1)\Gamma_e &= \frac{1}{2}(\gamma - 1)\Gamma_\rho u^2 + \gamma p u - \frac{4}{3}(\gamma - 1)\mu u \partial_x u \\ &= -\frac{1}{2}(\gamma + 1)\Gamma_\rho u^2 + \gamma \Gamma_p u + \frac{4}{3}\mu u \partial_x u \\ &= -\frac{1}{2}(\gamma + 1)\Gamma_\rho u_1^2 + \gamma \Gamma_p u_1 , \end{aligned} \quad (\text{SH.53})$$

where the final expression comes from evaluating the flux within the upstream region. This yields an equation for the derivative of the velocity

$$\frac{4}{3}\frac{\mu}{\Gamma_\rho} u \partial_x u = \frac{1}{2}(\gamma + 1)(u^2 - u_1^2) - \gamma \frac{\Gamma_p}{\Gamma_\rho}(u - u_1) . \quad (\text{SH.54})$$

Introducing the kinematic viscosity,  $\nu = \mu/\rho = \mu u/\Gamma_\rho$ , the equation becomes

$$\begin{aligned} \frac{4}{3}\nu \partial_x u &= \frac{1}{2}(\gamma + 1)(u - u_1) \left[ u + u_1 - \frac{2\gamma\Gamma_p}{(\gamma + 1)\Gamma_\rho} \right] \\ &= \frac{1}{2}(\gamma + 1)(u - u_1)(u - u_2) , \end{aligned} \quad (\text{SH.55})$$

where the downstream velocity is

$$u_2 = \frac{2\gamma\Gamma_p}{(\gamma + 1)\Gamma_\rho} - u_1 = \left[ \frac{\gamma - 1}{\gamma + 1} - \frac{2}{(\gamma + 1)M_1^2} \right] u_1 . \quad (\text{SH.56})$$

as can be verified from eq. (SH.35).

Equation (SH.55) shows that the velocity profile  $u(x)$  transitions between the two fixed points,  $u_2$  and  $u_1$  — these are the only two fixed points for velocity. Behavior in the vicinity of the fixed points is found by proposing small perturbations

$$u(x) = u_1 + \delta u_+(x) = u_2 - \delta u_-(x) .$$

Introducing these into eq. (SH.55) yields

$$\frac{4}{3}\nu \partial_x \delta u_\pm = \pm \frac{1}{2}(\gamma + 1)(u_1 - u_2) \delta u_\pm . \quad (\text{SH.57})$$

The perturbations grow or decay with a length scale

$$\lambda = \frac{8\nu}{3(\gamma + 1)|u_1 - u_2|} . \quad (\text{SH.58})$$

As  $x$  changes in the direction of the flow the downstream fixed point  $u_2$  will be an attractor *only* if  $|u_1| > |u_2|$ . This is another piece of evidence that shocks must produce a transition from supersonic to subsonic flow.

If we assume for simplicity that  $\nu$  is a constant then the solution to eq. (SH.55) is

$$u(x) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}|u_1 - u_2| \tanh(x/2\lambda) . \quad (\text{SH.59})$$

The hyperbolic tangent is essentially a smooth version of the step function. Rather than jumping discontinuously the velocity changes from its upstream value,  $u_1$ , to its downstream value,  $u_2$ , smoothly with a characteristic variation scale length

$$\lambda = \frac{8\nu}{3(\gamma + 1)|u_1 - u_2|} = \frac{4}{3} \frac{M_1}{M_1^2 - 1} \frac{\nu}{c_{s1}} , \quad (\text{SH.60})$$

after using eq. (SH.35). In the limit of ideal hydrodynamics,  $\nu \sim \mu \rightarrow 0$ , the shock becomes a genuine discontinuity,  $\lambda \rightarrow 0$ . Paradoxically, the specific entropy still jumps by the same amount, given by eq. (SH.39), even in this limit. Taking  $\mu \rightarrow 0$  does not result in  $\dot{Q} \rightarrow 0$ , since  $\partial_x u \sim u/\lambda \rightarrow \infty$  at the same time.<sup>5</sup>

Viscosity arises from a random walk of the particles composing the fluid. Motion at typical speed  $c_s$  with mean-free-path  $\ell_{\text{mfp}}$ , results in diffusion with coefficient  $D \sim c_s \ell_{\text{mfp}}$ . This includes the kinematic viscosity so  $\nu/c_s \sim \ell_{\text{mfp}}$  the mean free path. The shock's thickness is therefore comparable to the mean-free-path of the fluid particles.

Both extreme limits of expression (SH.60) are somewhat problematic. Shocks with extremely large Mach number are far thinner than a mean-free-path:  $\lambda \sim M_1^{-1} \ell_{\text{mfp}}$ . Fluid equations are not, however, physically valid on scales smaller than the mean-free-paths of their particles. It thus appears that strong shocks, i.e.  $M_1 \gg 1$ , cannot be treated using fluid equations. It is in this limit, however, that we cannot make the assumption that the kinematic viscosity  $\nu$  is constant. In reality  $\nu$  increases with temperature and thus the shock thickness remains greater than the mean-free-path.

The shock thickness in expression (SH.60) diverges in the opposite limit — a vanishingly weak shock,  $M_1 \rightarrow 1$ . This is the same limit in which the shock may be treated as a linear perturbation: a sound wave. Adding viscosity to eq. (SH.1) introduces diffusion leading to a front with thickness

$$\lambda \sim \sqrt{\nu t} , \quad (\text{SH.61})$$

growing with time. This behavior is standard in linear equations with diffusion. It is the steepening effect of the non-linearity that limits the shock thickness to the finite value of eq. (SH.60). As the non-linearity is taken to zero the thickness grows without bound. In any actual problem it would grow over time, and if the Mach number was sufficiently small it would fail to reach a steady state.

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<sup>5</sup>This is an example of a well-known situation where taking a limit,  $\mu \rightarrow 0$ , yields a result entirely different from setting  $\mu = 0$  at the outset.