An accretion disk is composed of ionized plasma \((\bar{m} = m_p/2)\) in an axisymmetric \((\partial/\partial \phi = 0)\), equilibrium \((\partial/\partial t = 0)\). Its fluid velocity is mostly azimuthal \((u \propto \hat{\phi})\), with a much smaller inward radial component (the accretion). Many accretion disks are treated in the thin disk limit, where axial distance \(z\) is assumed much smaller than the cylindrical radius \(\varpi\), and expansions are truncated at their lowest order in \(z/\varpi \ll 1\). The dominant, azimuthal flow velocity is written

\[
u = \varpi \Omega(\varpi) \hat{\phi},
\]

where \(\Omega(\varpi)\) is the angular rotation rate whose possible \(z\) dependence is neglected in the spirit of the thin disk. The gravitational force is only from the point-like central object, of mass \(M^*\), and is expanded in the thin disk limit

\[
\Psi(\varpi, z) = -\frac{GM_\ast}{\sqrt{\varpi^2 + z^2}} \simeq -\frac{GM_\ast}{\varpi} + \frac{GM_\ast}{2\varpi^3} z^2 = -\frac{GM_\ast}{\varpi} + \frac{1}{2} \Omega^2_{\text{kep}}(\varpi) z^2,
\]

where \(\Omega_{\text{kep}}\) is classical Keplerian rotation. The plasma temperature is also taken to depend on radius alone, \(T(\varpi)\).

a. Write down the axial \((\hat{z})\) component of the inviscid momentum equation using expressions (1) and (2). Use the fact that \(T\) is independent of \(z\) to find \(p(\varpi, z)\) within the disk. Show that the vertical scale height is \(h = c_{s,i}/\Omega_{\text{kep}}\), where \(c_{s,i} = \sqrt{k_B T/\bar{m}}\) is the isothermal sound speed. For an accreting object of \(M_\ast = 100 M_\odot\), and is expanded in the thin disk limit

\[
\rho(\varpi, 0) = \int_{-\infty}^{\infty} \rho(\varpi, z) dz,
\]

in terms of \(h(\varpi)\) and the equatorial mass density \(\rho(\varpi, 0)\).

b. Use the results of part a. write the disk’s surface density

\[
\Sigma(\varpi) = \int_{-\infty}^{\infty} \rho(\varpi, z) dz,
\]

in terms of \(h(\varpi)\) and the equatorial mass density \(\rho(\varpi, 0)\).

c. Write down the radial \((\hat{\varpi})\) component of the momentum equation evaluated in the equatorial plane, \(z = 0\). You must take care evaluating the advective term, \((u \cdot \nabla)\mathbf{u}\), using either curls and gradients or recalling that \(\partial \hat{\phi}/\partial \phi = -\hat{\varpi}\). Use scaling arguments to show that for a thin disk, i.e. \(h \ll \varpi\), then the pressure term is negligible and therefore the azimuthal velocity must have a Keplerian profile: \(\Omega(\varpi) \simeq \Omega_{\text{kep}}(\varpi)\). Does a small inward pressure gradient create faster or slower rotation?

d. The presence of a small radial flow component, \(u_{\varpi}(\varpi)\), will not affect the previous parts. Use it in the continuity equation to show that \(\varpi u_{\varpi} \Sigma\) is a constant — independent of \(\varpi\). Compute the mass flux, \(\dot{M}\), across a given radius — also known as the accretion rate.

e. Use the radial flow component in the azimuthal \((\hat{\phi})\) momentum equation evaluated in the equatorial plane, \(z = 0\). Keep only terms to leading order in \(u_{\varpi}/u_{\phi} \ll 1\). Here you must introduce viscosity. Use the simplified for form \(\mu \nabla^2 \mathbf{u}\), keep only leading order term (i.e. only \(u_{\phi}\)) and be (once again) careful to evaluate this correctly in the curvilinear coordinates. Find the accretion rate \(\dot{M}\) in terms of \(\mu\) and other properties of the disk. Use the viscosity for an
unmagnetized, fully-ionized plasma, given in Table 1.1 of the notes, to evaluate \( \dot{M} \) for the point on the disk used in part a. You can convert this to an accretion luminosity, \( \dot{M} c^2 \), to obtain something which can be easily compared to other objects visible in our universe — such as the Sun.

f. Assuming you found a ridiculously small accretion rate in part e., actual accretion cannot be driven by genuine viscosity. It must instead occur through \textit{turbulent viscosity}.\(^1\) This is exactly analogous to turbulent conductivity, and its diffusion rate scales exactly the same way

\[
\nu_{\text{turb}} \sim \tilde{\kappa}_{\text{turb}} \sim v_{\text{turb}}^2 \tau_{\text{cor}} .
\]

(4)

where \( \nu_{\text{turb}} \) is the root-mean-squared velocity of the turbulence and \( \tau_{\text{cor}} \) is its correlation time. In all likelihood, the turbulent velocity will scale with the isothermal sound speed, \( c_{s,i} \). Arguing along the same lines as for mixing-length theory, eddies will be comparable to the thickness of the disk, \( \ell_{\text{eddy}} \sim h \), and the correlation time will be the eddy-turnover time (the hydrodynamic time scale) \( \tau_{\text{cor}} \sim \ell_{\text{eddy}}/v_{\text{turb}} \sim h/c_{s,i} \). The upshot is a kinematic viscosity

\[
\nu_{\text{turb}} = \alpha c_{s,i} h ,
\]

(5)

where \( \alpha \) is a dimensionless parameter (\( \alpha \sim 1 \)) accounting for all the \( \sim s \) used in the foregoing argument. Use this in the expression you found in part e. to obtain an expression for the inward radial velocity \( u_\varpi \). Show that for a thin disk \( |u_\varpi| \ll u_\phi \) and \( |u_\varpi| \ll c_{s,i} \).

g. To provide a concrete example we choose a particular temperature distribution. In particular we take the isothermal speed to be

\[
c_{s,i}(\varpi) = \sqrt{\frac{k_B T(\varpi)}{m}} = \xi \sqrt{\frac{GM_*}{\varpi}} \varpi^{-1/2} .
\]

(6)

What is required of the dimensionless constant \( \xi \) in order that this describe a \textit{thin disk}? Use this to compute the time it takes for a fluid element starting at \( \varpi = R \) to fall onto the central mass. Express your result in terms of \( T_{\text{kep}}(R) \), the initial orbital period of the fluid element. Compute the \textit{trajectory} of this in falling mass, \( \varpi(\phi) \).

h. The disk can have stationary, spiral density perturbations from \textit{standing acoustic waves}. We solve for these in the eikonal limit, by proposing a phase function

\[
\varphi(\varpi, \phi) = m \left[ f(\varpi) - \phi \right] ,
\]

(7)

where \( m \) is the azimuthal mode number and \( f(\varpi) \) is a function we must find. An acoustic wave of frequency \( \omega \) must have a phase function satisfying the doppler-shifted dispersion relation

\[
\omega = u(\varpi) \cdot \nabla \varphi \pm c_{s,a}(\varpi) |\nabla \varphi| ,
\]

(8)

where \( c_{s,a} = \sqrt{\gamma c_{s,i}} \) is the adiabatic sound speed, and we continue to use the isothermal sound speed given in eq. (6). A standing wave will satisfy this relation with \( \omega = 0 \). Neglect doppler shifts from the radial flow and solve for \( f(\varpi) \). Use this to show that density ridges (\( \varphi = 0 \)) follow a spiral. Is this spiral tighter or looser than the accretion trajectory?

\(^1\)Or magnetic forces... or forces from a turbulent magnetic field, which take the form of turbulent viscosity.
Solution

a. The axial component of the momentum eq. is that of hydrostatic balance at a fixed radius \( \varpi \)

\[
\frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\partial \Psi}{\partial z} .
\]

(9)

Since \( T(\varpi) \) does not depend on \( z \) this is an isothermal atmosphere and we may immediately use eq. (2.12) from the notes

\[
p(\varpi, z) = p(\varpi, 0) \exp \left[ -\frac{\Psi(\varpi, z) - \Psi(\varpi, 0)}{c_{s,i}^2(\varpi)} \right] = p(\varpi, 0) \exp \left[ -\frac{\Omega_{\text{kep}}^2(\varpi) z^2}{2 c_{s,i}^2(\varpi)} \right],
\]

(10)

where \( c_{s,i} = \sqrt{\frac{k_B T}{\bar{m}}} \) is the isothermal sound speed, and we have used eq. (2). The pressure and density both, therefore, have Gaussian vertical structure

\[
p(\varpi, z) = p(\varpi, 0) e^{-z^2/2h^2(\varpi)}, \quad \rho(\varpi, z) = \rho(\varpi, 0) e^{-z^2/2h^2(\varpi)},
\]

(11)

with the scale height

\[
h(\varpi) = \frac{c_{s,i}(\varpi)}{\Omega_{\text{kep}}(\varpi)} = \sqrt{\frac{k_B T(\varpi)}{GmM_*}} \varpi^{3/2} .
\]

(12)

One AU from a central object of 100 solar masses, the orbital period will be one tenth of a year: \( 3 \times 10^6 \) sec. This means \( \Omega_{\text{kep}} = 2 \times 10^{-6} \) sec. Using the mean mass \( \bar{m} = m_p/2 = 0.8 \times 10^{-24} \) g, gives an isothermal sound speed \( c_{s,i} = \sqrt{k_B T / \bar{m}} \). The scale height of the disk at this point is \( h = c_{s,i}/\Omega_{\text{kep}} = 6.4 \times 10^{11} \) cm. This is 0.04 AU, or 4% of the radius. This disk is indeed thin.

b. Using the Gaussian profile in the definition of surface density, (3), gives

\[
\Sigma(\varpi) = \rho(\varpi, 0) \int_{-\infty}^{\infty} e^{-z^2/2h^2(\varpi)} dz = \sqrt{2\pi} \rho(\varpi, 0) h(\varpi) .
\]

(13)

c. The advective term in the momentum equation is

\[
(u \cdot \nabla)u = \frac{u_\varpi}{\varpi} \frac{\partial (u_\varphi \hat{\varphi})}{\partial \varphi} = \frac{u_\varpi^2}{\varpi} \frac{\partial \hat{\varphi}}{\partial \varphi} = -\frac{u_\varpi^2}{\varpi} \hat{\varpi} = -\varpi \Omega^2(\varpi) \hat{\varpi} .
\]

(14)

Alternatively one may use vector identities

\[
(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2} \nabla |u|^2 = \left[ \frac{1}{\varpi} \frac{\partial (\varpi u_\varphi)}{\partial \varpi} \right] \times u_\varphi \hat{\varphi} + \frac{1}{2} \nabla u_\varphi^2 \\
= \left[ -\frac{1}{\varpi} \frac{\partial (\varpi u_\varphi)}{\partial \varpi} + \frac{1}{\varpi} \frac{\partial u_\varphi}{\partial \varpi} \right] \hat{\varpi} = -\frac{u_\varphi^2}{\varpi} \hat{\varpi} = -\varpi \Omega^2(\varpi) \hat{\varpi} ,
\]

to arrive at the same result (but more laboriously).

The radial component of the gravitational contribution is

\[
-\frac{\partial \Psi}{\partial \varpi} \bigg|_{z=0} = \frac{\partial}{\partial \varpi} \left( \frac{GmM_*}{\varpi} \right) = -\frac{GM_*}{\varpi^2} = -\varpi \Omega_{\text{kep}}^2(\varpi) ,
\]

(15)
after using eq. (2) at \( z = 0 \). Combining these into the radial component of the entire momentum equation yields

\[
-\varpi \Omega^2(\varpi) = -\frac{1}{\rho(\varpi, 0)} \frac{\partial p}{\partial \varpi} - \varpi \Omega^2_{\text{kep}}(\varpi) .
\] (16)

Taking the ratio of scales of the first and second terms on the right hand side yields

\[
\frac{\rho^{-1}(\partial p/ \partial \varpi)}{\varpi \Omega^2_{\text{kep}}} \sim \frac{p/\rho}{\varpi^2 \Omega^2_{\text{kep}}} = \frac{c^2_{s,i}}{\varpi^2 \Omega^2_{\text{kep}}} = \frac{h^2}{\varpi^2} ,
\] (17)

after using the disk scale height \( h = c_{s,i}/\Omega_{\text{kep}} \). The pressure contribution is smaller than the gravitational contribution by \( (h/\varpi)^2 \ll 1 \). After dropping the pressure term the momentum equation

\[
-\varpi \Omega^2(\varpi) \simeq -\varpi \Omega^2_{\text{kep}}(\varpi) ,
\] (18)

is clearly satisfied by \( \Omega(\varpi) = \pm \Omega_{\text{kep}}(\varpi) \). We adopt the upper sign so the rotation is counterclockwise for convenience. Of course a clockwise disk is an equally valid solution — but just seems backwards somehow.

Assuming \( |\Omega - \Omega_{\text{kep}}| \ll \Omega_{\text{kep}} \) we can solve for the difference

\[
\Omega^2_{\text{kep}} - \Omega^2 \simeq 2\Omega_{\text{kep}} (\Omega_{\text{kep}} - \Omega) = -\frac{1}{\varpi \rho} \frac{\partial p}{\partial \varpi} = -\frac{p}{\varpi \rho} \frac{\partial \ln(p)}{\partial \varpi} = -\frac{c^2_{s,i}}{\varpi} \frac{\partial \ln(p)}{\partial \varpi} .
\] (19)

This provides the angular rotation

\[
\Omega(\varpi) = \Omega_{\text{kep}}(\varpi) + \frac{c^2_{s,i}}{2\varpi \Omega_{\text{kep}}} \frac{\partial \ln(p)}{\partial \varpi} = \Omega_{\text{kep}}(\varpi) \left[ 1 + \frac{h^2}{2\varpi} \frac{\partial \ln(p)}{\partial \varpi} \right] .
\] (20)

For an inward-directed pressure gradient, \( \partial \ln(p)/\partial \varpi < 0 \) and \( \Omega < \Omega_{\text{kep}} \) — the flow is sub-Keplerian. Any large, solid body in this disk would thus feel a backwards drag, which would ultimately pull the object inward.

d. The steady continuity equation is

\[
0 = \nabla \cdot (\rho u) = \frac{1}{\varpi} \frac{\partial (\varpi \rho u_\varpi)}{\partial \varpi} + \frac{1}{\varpi} \frac{\partial (\rho u_\phi)}{\partial \phi} = \frac{1}{\varpi} \frac{\partial (\varpi \rho u_\varpi)}{\partial \varpi} ,
\] (21)

after invoking axisymmetry to drop the \( \partial / \partial \phi \) term. This shows that \( \varpi \rho u_\varpi = F(z) \) is independent of \( \varpi \), but will depend on \( z \). We can integrate this \( z \)-dependent function to obtain

\[
\int_{-\infty}^{\infty} F(z) \, dz = \int_{-\infty}^{\infty} \varpi \rho(\varpi, z) u_\varpi(\varpi) \, dz = \varpi u_\varpi(\varpi) \int_{-\infty}^{\infty} \rho(\varpi, z) \, dz = \varpi u_\varpi(\varpi) \Sigma(\varpi) ,
\] (22)

after using the definition of surface density eq. (3).

To obtain the inward mass flux we integrate over a cylindrical surface with an inward directed surface normal

\[
M = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \rho u \cdot \left( -\frac{d}{d\varpi} \varpi d\phi \right) \, dz = -2\pi \int_{-\infty}^{\infty} dz \varpi \rho(\varpi, z) u_\varpi(\varpi)
\]

\[= -2\pi \varpi u_\varpi(\varpi) \Sigma(\varpi) .
\] (23)
Provided the flow is inward, $u_\varpi < 0$, this will be a positive quantity, $\dot{M} > 0$.

e. The leading order azimuthal terms in the advective derivative are

$$\hat{\phi} \cdot \left[ (u \cdot \nabla) u \right] = (u \cdot \nabla)(\hat{\phi} \cdot u) - u \cdot \left[ (u \cdot \nabla) \hat{\phi} \right] = u \cdot \left[ \hat{\omega} \frac{\partial u_\varphi}{\partial \varpi} \right] - u \cdot \left[ \frac{u_\varphi}{\varpi} \frac{\partial \hat{\phi}}{\partial \varpi} \right]$$

$$= u_\varpi \frac{\partial (\varpi \Omega)}{\partial \varpi} - u \cdot \left[ \frac{u_\varphi}{\varpi} (- \hat{\omega}) \right] = u_\varpi \frac{\partial (\varpi \Omega)}{\partial \varpi} + u_\varpi \Omega . \quad (24)$$

Using the fact that $\Omega = \Omega_{\text{kep}} \propto \varpi^{-3/2}$ we find

$$\hat{\phi} \cdot \left[ (u \cdot \nabla) u \right] = \frac{1}{2} \Omega_{\text{kep}} u_\varpi . \quad (25)$$

To evaluate the Laplacian of a vector field in curvilinear coordinates we must use the vector identity

$$\nabla^2 u = \nabla (\nabla \cdot u) - \nabla \times (\nabla \times u) = - \nabla \times (\nabla \times u) ,$$

since the divergence of the leading order velocity, eq. (1), vanishes. Taking the first curl gives

$$\nabla \times u = \nabla \times (\varpi \Omega \hat{\phi}) = \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 \Omega) \hat{z} . \quad (26)$$

The second then gives us

$$\nabla^2 u = \frac{\partial}{\partial \varpi} \left[ \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 \Omega) \right] \hat{\phi} . \quad (27)$$

Using the fact that $\Omega = \Omega_{\text{kep}} \propto \varpi^{-3/2}$ we find

$$\nabla^2 u = - \frac{3}{4} \frac{\Omega_{\text{kep}}}{\varpi} \hat{\phi} . \quad (28)$$

Since neither the pressure nor gravity contribute azimuthal components of the momentum equation we have

$$\frac{1}{2} \rho(\varpi,0) \Omega_{\text{kep}} u_\varpi = - \mu \frac{3}{4} \frac{\Omega_{\text{kep}}}{\varpi} . \quad (29)$$

Solving for $u_\varpi$ yields

$$u_\varpi = - \frac{3}{2} \frac{\mu}{\rho(\varpi,0)} \frac{1}{\varpi} = - \frac{3}{2} \frac{\mu}{\varpi} \frac{\sqrt{2\pi}}{\Sigma} . \quad (30)$$

after using (13) to replace $\rho(\varpi,0)$. Using this in eq. (23) gives an accretion rate

$$\dot{M} = - 2\pi \varpi \Sigma u_\varpi = 3\pi \sqrt{2\pi} \mu \varpi \Sigma . \quad (31)$$

The mass accretion rate is simply proportional to dynamics viscosity $\mu$.

According to Table 1.1, the dynamic viscosity of an unmagnetized plasma at $T = 10^4$ K (i.e. $T_6 = 0.01$) is $\mu = 1.2 \times 10^{-6}$ g/cm/s. Multiplying this by $h = 6.4 \times 10^{11}$ cm, from part a., yields an accretion rate

$$\dot{M} = 3\pi \sqrt{2\pi} \times 7.7 \times 10^5 \text{ g/s} = 1.8 \times 10^7 \text{ g/s} = \frac{M_\odot}{10^{26} \text{ sec}} . \quad (32)$$

The accretion luminosity of this would be

$$\dot{M} c^2 = 1.6 \times 10^{28} \text{ erg/s} = 4 \times 10^{-6} L_\odot . \quad (33)$$
This is ridiculously small for such an object. It has a bolometric magnitude of $M_{\text{bol}} = 18$, meaning it would be (barely) visible with a 4m telescope if it were 25 parsecs away and 100% of its accretion luminosity were converted to visible light.

f. Substituting expression (5) into eq. (30) we find

$$u_{\varpi} = -\frac{3}{2} \mu \rho(\varpi,0) \frac{1}{\varpi} = -\frac{3}{2} \frac{\nu}{\varpi} = -\frac{3}{2} \alpha c_{s,i} \frac{h}{\varpi} ,$$

after using $\mu = \rho \nu$, and evaluating this at the equatorial plane.

Dividing this by the azimuthal velocity, found in part c., to be Keplerian, gives

$$\frac{u_{\varpi}}{u_{\phi}} = -\frac{3}{2} \alpha \frac{c_{s,i} h}{\Omega_{\text{kep}} \varpi^2} = -\frac{3}{2} \alpha \frac{h^2}{\varpi^2} .$$

This is clearly small provided $h/\varpi \ll 1$, and $\alpha \sim 1$ as stated. We also get

$$\frac{u_{\varpi}}{c_{s,i}} = -\frac{3}{2} \alpha \frac{h}{\varpi}$$

which is also $\ll 1$ so the inward flow is subsonic provided again the disk is thin.

g. Using eq. (6) in eq. (12) gives a scale height

$$h(\varpi) = \frac{c_{s,i}}{\Omega_{\text{kep}}(\varpi)} = \frac{\xi \sqrt{GM_* \varpi^{-1/2}}}{\sqrt{GM_* \varpi^{-3/2}}} = \xi \varpi .$$

Clearly for $h \ll \varpi$ we require $\xi \ll 1$. If this is true the disk will be thin everywhere.

Note that if we had chosen $c_{s,i} \propto \varpi^{-s/2}$, i.e. $T \propto \varpi^s$, the ratio $h/\varpi \propto \varpi^{(1-s)/2}$. Thus for $s < 1$ the disk would fail to be thin at sufficiently large radii and for $s > 1$ it would fail to be thin at sufficiently small radii. Only for the choice $c_{s,i} \propto \varpi^{-1/2}$ is the disk thin everywhere (or nowhere in the case $\xi \gg 1$).

Using eq. (6) in eq. (34) gives the inward velocity

$$u_{\varpi} = -\frac{3}{2} \alpha c_{s,i} \frac{h}{\varpi} = -\frac{3}{2} \alpha \xi^2 \sqrt{GM_* \varpi^{-1/2}} .$$

Equating this with the radial velocity of a fluid element $d\varpi/dt$ yields

$$\varpi^{1/2} \frac{d\varpi}{dt} = -\frac{3}{2} \alpha \xi^2 \sqrt{GM_*} ,$$

whose solution is

$$R^{3/2} - \varpi^{3/2} = \frac{9}{4} \alpha \xi^2 \sqrt{GM_*} t .$$

The central object is reached ($\varpi \to 0$) in finite time

$$t_{\text{fall}}(R) = \frac{4}{9} \frac{R^{3/2}}{\alpha \xi^2 \sqrt{GM_*}} = \frac{4}{9} \frac{1}{\alpha \xi^2 \Omega_{\text{kep}}(R)} = \frac{2}{9\pi} \frac{T_{\text{kep}}(R)}{\alpha \xi^2} .$$

Since $\xi \ll 1$ this time is much longer than the orbital time.

We find the trajectory from the equation

$$\frac{d\varpi}{d\phi} = \frac{d\varpi}{dt} \frac{dt}{d\phi} = \frac{u_{\varpi}}{u_{\phi}} = -\frac{3}{2} \alpha \frac{h^2}{\varpi} = -\frac{3}{2} \alpha \xi^2 \varpi ,$$

(41)
after using equation (35) and \( h = \xi \varpi \) from eq. (36). The accreting trajectory is an exponential spiral

\[
\varpi_{\text{ac}}(\phi) = R \exp \left[ -\frac{3}{2} \alpha \xi^2 \phi \right].
\] (42)

This is shown in fig. 1 for a disk with \( \xi = 0.1 \) and \( \alpha = 1 \). It is typical to choose \( \alpha \approx 0.1 \), but doing so would produce a spiral too tight to clearly show — the same 25 cycles would have ended at \( \varpi = 0.8R \) rather than \( \varpi = 0.1R \).

Figure 1: The structure of a disk with \( \xi = 0.1 \) and \( \alpha = 1 \). The red spiral curve is the trajectory of a fluid element, \( \varpi_{\text{ac}}(\phi) \), given by eq. (42). For clarity only 25 cycles of the spiral are plotted, ending therefore at \( \varpi \approx 0.1R \). Grey scale is density, perturbed by a standing acoustic mode (see part h.). The blue curve shows the spiral curve \( \varpi_{\text{ds}}(\phi) \) given by eq. (49) with \( \gamma = 5/3 \). This is the center of the density perturbation \( \varphi = 0 \).

h. Setting \( \omega = 0 \) in eq. (8), and dropping \( u_\varpi \), yields

\[
\omega = 0 = \frac{u_\phi}{\varpi} \frac{\partial \phi}{\partial \phi} \pm c_{s,0}(\varpi) |\nabla \varphi| = -m \frac{\sqrt{G M}}{\varpi^{3/2}} \pm \sqrt{7} \xi \sqrt{G M}^{1/2} \varpi^{-1/2} |\nabla \varphi|,
\] (43)

after using eqs. (6) and (7). Rearranging gives the form

\[
\frac{m}{\sqrt{7} \xi} = \pm \varpi |\nabla \varphi|.
\] (44)
Clearly the upper sign goes with \( m > 0 \) and the lower sign with \( m < 0 \). Squaring this
\[
\omega^2 \left( \frac{df}{d\omega} \right)^2 = \frac{1}{\gamma \xi^2} - 1 \approx \frac{1}{\gamma \xi^2} ,
\]
after using the fact that \( \xi \ll 1 \). This yields the equation
\[
\frac{df}{d\omega} = \pm \frac{1}{\sqrt{\gamma \xi}} \frac{1}{\omega} ,
\]
where the signs here are not related to the ones eq. (8). Its solution is
\[
f(\omega) = \pm \frac{1}{\sqrt{\gamma \xi}} \ln \left( \frac{\omega}{R} \right) ,
\]
after choosing to make \( f(R) = 0 \). Note that \( m \) does not appear in the function, so it seems all mode numbers have the same phase structure.

A density ridge will fall along the curve where \( \varphi(\omega, \phi) = 0 \). In other words where
\[
\phi = f(\omega) = \pm \frac{1}{\sqrt{\gamma \xi}} \ln \left( \frac{\omega}{R} \right) .
\]
Inverting this yields an exponential spiral
\[
\omega_{ds}(\phi) = R \exp \left[ \pm \sqrt{\gamma \xi} \phi \right] ,
\]
describing the shape of the stationary density spiral. The upper and lower signs give spirals which wind outward and inward when followed in the counterclockwise sense. The inward spiral (lower sign) resembles the spiraling trajectory of a fluid element, eq. (42). The accretion spiral is more tightly wound (see fig. 1) provided \( \xi \ll \sqrt{4\gamma/9\alpha^2} = \sqrt{20/27\alpha^2} \). Indeed, in the absence of accretion, \( \alpha \to 0 \), the fluid elements orbit in circles but the density perturbation retains its same spiral structure (recall we took \( u_\varpi = 0 \) in deriving it). This standing spiral structure is reminiscent of spiral arms in galaxies, which represent a standing wave mode and not the motion of material elements like stars.

The inward and outward spirals are related to waves with inward and outward group velocities. The radial wavenumber of each is found from the radial component of \( \nabla \varphi \)
\[
k_{\varpi} = \frac{\partial \varphi}{\partial \varpi} = m \frac{df}{d\omega} = \pm m \frac{1}{\sqrt{\gamma \xi}} \frac{1}{\omega} ,
\]
where the \( \pm \) gives from eq. (46), which has been substituted for \( df/d\omega \). The radial component of the group velocity is
\[
v_{g,\varpi} = \frac{\partial \omega}{\partial k_{\varpi}} = s c_{s,\alpha} \frac{k_{\varpi}}{|\nabla \varphi|} = \pm (sm) \frac{1}{\sqrt{\gamma \xi}} \frac{1}{\omega} ,
\]
where \( s = \pm 1 \) is the sign from the dispersion relation (8). We remarked above that \( sm = |m| > 0 \) since the sign of \( m \) had to match this choice of sign. Thus the upper and lower sign in (49) corresponds to wave with outward and inward group velocity. If the perturbation were initiated from the outside then \( v_{g,\varpi} < 0 \) and only the inner sign would appear. That means the density spiral would move inward when traced counterclockwise (i.e. with the flow).