

Non-thermal particles

Stochastic Acceleration

Lecture 19

April 3, 2017

Fokker-Planck eq: what it means

$f_\sigma(\mathbf{x}, \mathbf{v})$ = phase space density of species σ ($= e, p$)
averaged over ensemble of particle configurations

$$\mathbf{F}(\mathbf{x}, \mathbf{v}) = q_\sigma \mathbf{E}(\mathbf{x}) + q_\sigma \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{x}) \quad \text{Force felt by each particle in ensemble @ } (\mathbf{x}, \mathbf{v})$$

ensemble-
averaged
fields

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}(\mathbf{x}) = \frac{4\pi e}{c} \int [\mathbf{v} f_p(\mathbf{x}, \mathbf{v}) - \mathbf{v} f_e(\mathbf{x}, \mathbf{v})] d^3v$$

& similarly for $\mathbf{E}(\mathbf{x})$

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\partial}{\partial x_i} (v_i f_\sigma) - \frac{\partial}{\partial v_i} \left[\frac{F_i(\mathbf{x}, \mathbf{v})}{m} f_\sigma \right] - \frac{\partial}{\partial v_i} \left(\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle f_\sigma \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle f_\sigma \right)$$

Mean effects from fluctuating forces –
averaged over ensemble

One use for F-P eqn. linear plasma waves

1. Start w/ static, uniform **solution** to F-P: e.g. Maxwellians

$$f_{\sigma}^{(0)}(\mathbf{v}) \quad \mathbf{E}^{(0)} = 0 \quad \mathbf{B}^{(0)} = B_0 \hat{\mathbf{z}}$$

2. Add small, plane-wave perturbation

$$f_{\sigma}(\mathbf{x}, \mathbf{v}, t) = f_{\sigma}^{(0)}(\mathbf{v}) + \hat{f}_{\sigma}^{(1)}(\mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

$$\mathbf{E}(\mathbf{x}, t) = \hat{\mathbf{E}}^{(1)} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \quad \text{etc.}$$

3. Expand F-P to linear order in perturbation

$$-i\omega \hat{f}_{\sigma}^{(1)} = -i(\mathbf{k} \cdot \mathbf{v}) \hat{f}_{\sigma}^{(1)} - \frac{q_{\sigma}}{m_{\sigma} c} (\mathbf{v} \times B_0 \hat{\mathbf{z}}) \cdot \frac{\partial \hat{f}_{\sigma}^{(1)}}{\partial \mathbf{v}} - \frac{q_{\sigma}}{m_{\sigma}} \frac{\partial f_{\sigma}^{(0)}}{\partial \mathbf{v}} \cdot \left(\hat{\mathbf{E}}^{(1)} + \frac{\mathbf{v}}{c} \times \hat{\mathbf{B}}^{(1)} \right) - \dots$$

$$\hat{\mathbf{B}}^{(1)} = i \frac{4\pi e}{ck^2} \mathbf{k} \times \int \left[\mathbf{v} \hat{f}_p^{(1)}(\mathbf{v}) - \mathbf{v} \hat{f}_e^{(1)}(\mathbf{v}) \right] d^3 v \quad \& \text{ similarly for } \hat{\mathbf{E}}^{(1)}$$

linear integro-differential equation

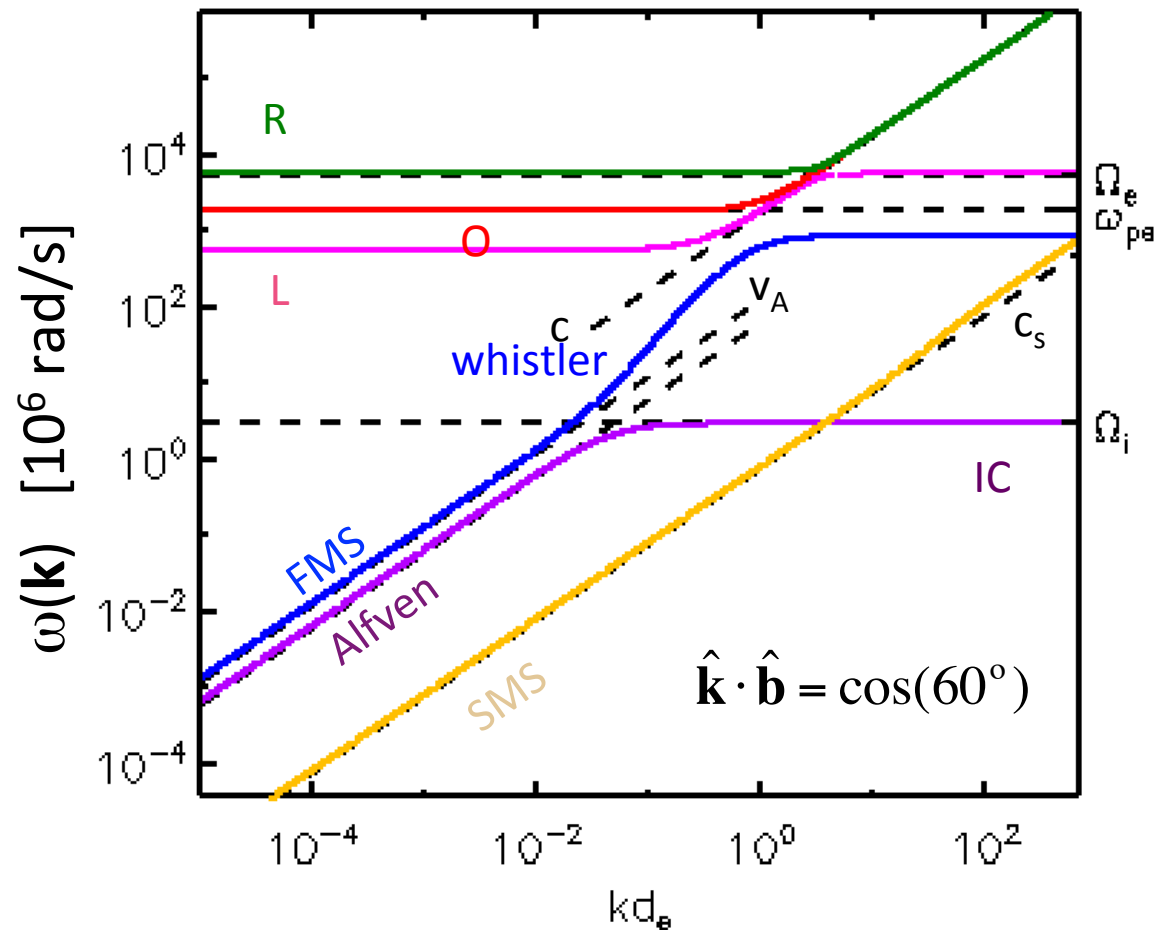
4. Solve linearized eqn. for particular \mathbf{k}

Find many solutions:

- eigenfunctions... $\hat{f}_e^{(1)}(\mathbf{v})$ $\hat{f}_p^{(1)}(\mathbf{v})$
- ... eigenvectors ... $\hat{\mathbf{E}}^{(1)}$ $\hat{\mathbf{B}}^{(1)}$ (transverse, longitudinal, ...)
- ... and eigenfrequencies $\omega(\mathbf{k})$ – wave modes
- e-freqs complex –
 - $\text{Im}(\omega) < 0$: damped
 - $\text{Im}(\omega) > 0$: unstable
- all dependant on choice of equilibrium

$$f_{\sigma}^{(0)}(\mathbf{v})$$

$$\mathbf{B}^{(0)} = B_0 \hat{\mathbf{z}}$$



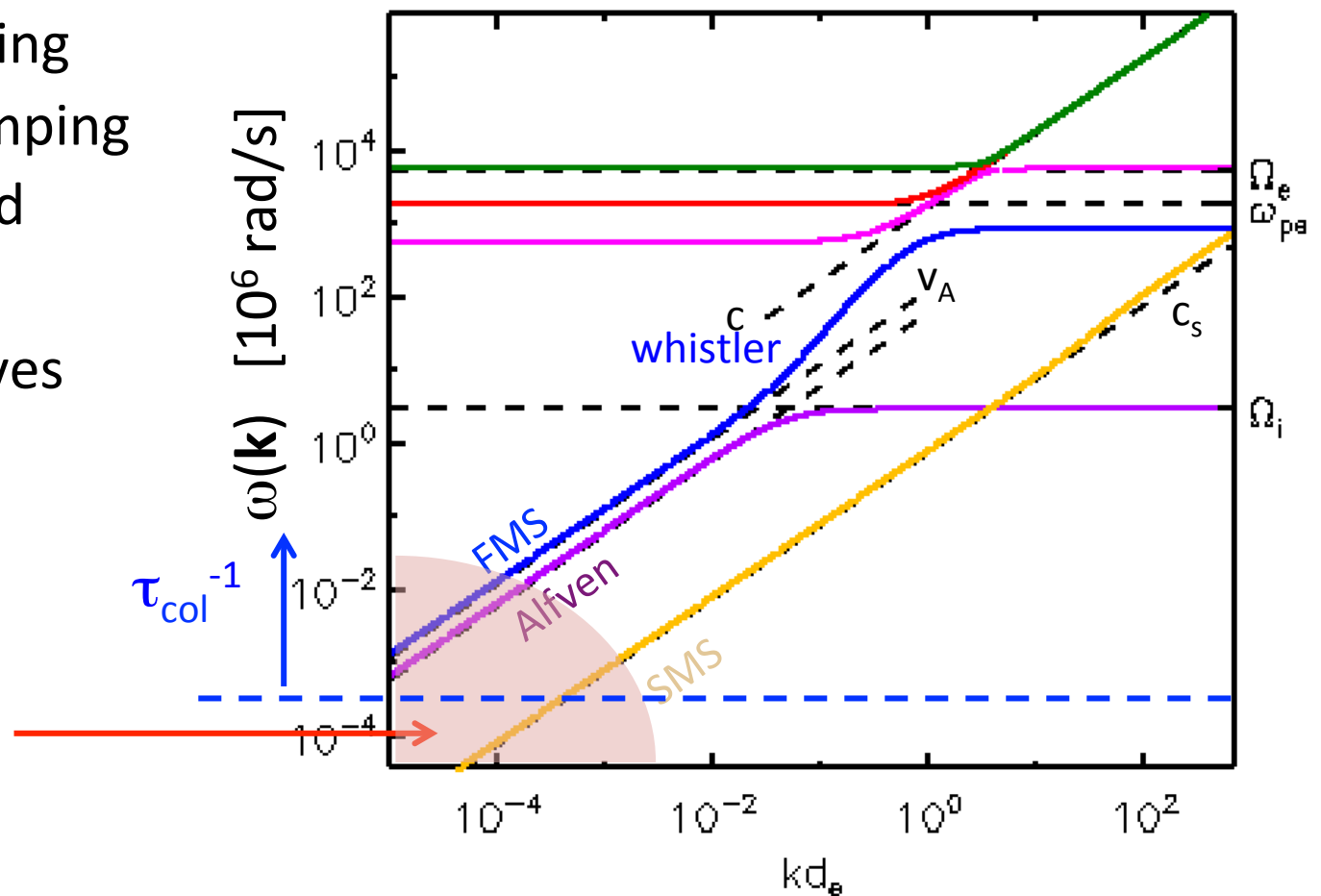
For many solutions (waves): $|\omega(\mathbf{k})| \gg \tau_{\text{col}}^{-1} = A(v)/v$

→ can neglect collisions i.e. discard $A_i(\mathbf{v})$ & $D_{ij}(\mathbf{v})$

solve collisionless F-P eqn. *a.k.a.* **Vlasov Eqn.**

- initial solution, $f_{\sigma}^{(0)}(\mathbf{v})$, need not be Maxwellian – many distributions work
- still find damping
i.e. Landau damping
- sometimes find instabilities
→ growing waves

**MHD –
small ω ,
small k**



New meaning for Fokker-Planck

\exists many random linear waves = small random $\mathbf{E}(\mathbf{x},t)$ & $\mathbf{B}(\mathbf{x},t)$

$f_\sigma(\mathbf{x}, \mathbf{v})$ = phase space density of species σ ($= e, p$)

averaged over ensemble of **particle configurations**

AND ensemble of **random waves**

ensemble-averaged fields

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}(\mathbf{x}) = \frac{4\pi e}{c} \int [\mathbf{v} f_p(\mathbf{x}, \mathbf{v}) - \mathbf{v} f_e(\mathbf{x}, \mathbf{v})] d^3 \mathbf{v}$$

WILL NOT include the waves

& similarly for $\mathbf{E}(\mathbf{x})$

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\partial}{\partial x_i} (v_i f_\sigma) - \frac{\partial}{\partial v_i} \left[\frac{F_i(\mathbf{x}, \mathbf{v})}{m} f_\sigma \right] - \frac{\partial}{\partial v_i} \left(\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle f_\sigma \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle f_\sigma \right)$$

Mean effects from fluctuating forces – **averaged over particle ensemble AND wave ensemble**

New meaning for Fokker-Planck

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\partial}{\partial x_i} (v_i f_\sigma) - \frac{\partial}{\partial v_i} \left[\frac{F_i(\mathbf{x}, \mathbf{v})}{m} f_\sigma \right] - \frac{\partial}{\partial v_i} \left(\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle f_\sigma \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle f_\sigma \right)$$

diffusion includes
2 components

$$\frac{1}{2} \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle = D_{ij} = D_{ij}^{(\text{col})} + D_{ij}^{(\text{turb})}$$

effect of random interaction
between discrete particles –
collisions (as before)

$$D_{ij}^{(\text{col})}(\mathbf{v}) \sim \frac{4\pi e^4 \Lambda n' v_{rms}'^2}{m^2} \frac{\delta_{ij}}{v^3}$$

effect of random forces from
turbulence of plasma waves
new element

spectral density of
plasma waves

$$D_{ij}^{(\text{turb})}(\mathbf{v}) \sim \frac{\pi}{2} \frac{e^2}{m^2} \int d^3k d\omega \overbrace{W_{ij}(\mathbf{k}, \omega)} \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

New meaning for Fokker-Planck

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\partial}{\partial x_i}(v_i f_\sigma) - \frac{\partial}{\partial v_i} \left[\frac{F_i(\mathbf{x}, \mathbf{v})}{m} f_\sigma \right] - \frac{\partial}{\partial v_i} \left(\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle f_\sigma \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle f_\sigma \right)$$

diffusion includes
2 components

$$\frac{1}{2} \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle = D_{ij} = D_{ij}^{(\text{col})} + D_{ij}^{(\text{turb})}$$

Energy

$$\varepsilon_\sigma = \frac{1}{2} m_\sigma \int v^2 f_\sigma(\mathbf{x}, \mathbf{v}) d^3 v$$

Saw in last lecture that $D_{ij}^{(\text{col})}$

from elastic collisions will conserve particle energy.

Q: Do we expect $D_{ij}^{(\text{turb})}$ to conserve particle energy as well? Decrease it? Increase it?

Illustration of turbulent effect

Simplest case: unmagnetized plasma $\mathbf{B}^{(0)} = 0$
(but not most relevant)

largest force from
random electric field

$$E_i(\mathbf{x}, t) = \int \underbrace{\hat{E}_i(\mathbf{k}, \omega)}_{\text{random Fourier coefficients}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} d^3k d\omega$$

random Fourier coefficients

Ensemble of random fields:

- Fourier coefficients $\hat{E}_i(\mathbf{k}, \omega)$ are random complex variables

- Zero mean $\langle \hat{E}_i(\mathbf{k}, \omega) \rangle = 0$

- Variance specified by spectrum $W_{ij}(\mathbf{k}, \omega)$

$$E_i(\mathbf{x}, t) = \int \hat{E}_i(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} d^3k d\omega$$

$$\langle E_i(\mathbf{x}, t) E_j(\mathbf{x}, t) \rangle = \int d^3k d\omega \int d^3k' d\omega' e^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{x} - i(\omega + \omega') t} \langle E_i(\mathbf{k}, \omega) E_j(\mathbf{k}', \omega') \rangle$$

Stationary homogeneous turbulence \rightarrow
 correlation only between $\hat{E}_i(\mathbf{k}, \omega)$ and $\hat{E}_j(-\mathbf{k}, -\omega)$

$$\langle \hat{E}_i(\mathbf{k}, \omega) \hat{E}_j(\mathbf{k}', \omega') \rangle = \underbrace{W_{ij}(\mathbf{k}, \omega)}_{\substack{\text{spectral} \\ \text{density}}} \underbrace{\delta(\mathbf{k} + \mathbf{k}')}_{\substack{\text{homo-} \\ \text{geneous}}} \underbrace{\delta(\omega + \omega')}_{\substack{\text{stationary}}}$$

avgs. indep. of \mathbf{x} or t

$$\langle E_i(\mathbf{x}, t) E_j(\mathbf{x}, t) \rangle = \int d^3k d\omega W_{ij}(\mathbf{k}, \omega) = \langle E_i E_j \rangle$$

energy density in turbulent waves:


$$\varepsilon_{\text{turb}} = \frac{1}{8\pi} \langle E_i E_i \rangle = \frac{1}{8\pi} \int d^3k d\omega W_{ii}(\mathbf{k}, \omega)$$

velocity change
due to waves

$$\Delta v_i = \int_0^{\Delta t} \dot{v}_i(t) dt = \int_0^{\Delta t} \frac{q}{m} E_i dt = \frac{q}{m} \int_0^{\Delta t} E_i(\mathbf{v}t, t) dt$$

using $E_i(\mathbf{v}t, t) = \int \hat{E}_i(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)t} d^3k d\omega$

motion of free
particle (unaffected
by waves)



$$\langle \Delta v_i \Delta v_j \rangle = \frac{q^2}{m^2} \int_0^{\Delta t} dt \int_0^{\Delta t} dt' \int d^3k d\omega \int d^3k' d\omega' e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)t} e^{i(\mathbf{k}' \cdot \mathbf{v} - \omega')t'} \langle \hat{E}_i(\mathbf{k}, \omega) \hat{E}_j(\mathbf{k}', \omega') \rangle$$

and using $\langle \hat{E}_i(\mathbf{k}, \omega) \hat{E}_j(\mathbf{k}', \omega') \rangle = W_{ij}(\mathbf{k}, \omega) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega')$

yields
correlation

$$\langle \Delta v_i \Delta v_j \rangle = \frac{q^2}{m^2} \int d^3k d\omega W_{ij}(\mathbf{k}, \omega) \left[\int_0^{\Delta t} dt \int_0^{\Delta t} dt' e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)(t-t')} \right]$$

what is this?

assuming $\Delta t \gg$
wave periods

$$\int_0^{\Delta t} dt' e^{i(\omega - \mathbf{k} \cdot \mathbf{v})t'} \approx \int_0^{\infty} dt' e^{i(\omega - \mathbf{k} \cdot \mathbf{v})t'} = \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

$$\rightarrow \int_0^{\Delta t} dt \int_0^{\Delta t} dt' e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)(t-t')} = \int_0^{\Delta t} dt \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}) = \Delta t \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

$$\begin{aligned} \rightarrow \langle \Delta v_i \Delta v_j \rangle &= \frac{e^2}{m^2} \int d^3k d\omega W_{ij}(\mathbf{k}, \omega) \left[\int_0^{\Delta t} dt \int_0^{\Delta t} dt' e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)(t-t')} \right] \\ &= \Delta t \pi \frac{e^2}{m^2} \int d^3k d\omega W_{ij}(\mathbf{k}, \omega) \underbrace{\delta(\omega - \mathbf{k} \cdot \mathbf{v})} \end{aligned}$$

Doppler resonance: $\omega' = 0$

Fokker-Planck coefficient

$$D_{ij}^{(\text{turb})}(\mathbf{v}) = \frac{1}{2} \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle = \frac{\pi}{2} \frac{e^2}{m^2} \int d^3k d\omega \underbrace{W_{ij}(\mathbf{k}, \omega)}_{\text{spectral density}} \underbrace{\delta(\omega - \mathbf{k} \cdot \mathbf{v})}_{\text{in resonant waves}}$$

spectral density in resonant waves

$$D_{ij}^{(\text{turb})}(\mathbf{v}) = \frac{\pi}{2} \frac{e^2}{m^2} \int d^3k d\omega W_{ij}(\mathbf{k}, \omega) \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

Can express
diffusion in
form

$$D_{ij}^{(\text{turb})}(\mathbf{v}) = 4\pi^2 \frac{e^2}{m^2} \epsilon_{\text{turb}} \tau_{ij}(\mathbf{v})$$

\propto to energy density
of turbulence

\mathbf{v} dependence from
resonance in spectral
shape alone

$$\tau_{ij}(\mathbf{v}) \equiv \frac{\int d^3k W_{ij}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})}{\int d^3k d\omega W_{nn}(\mathbf{k}, \omega)}$$

Turbulent Diffusion – more relevant examples

Lower-hybrid waves in a
magnetized plasma
(Moghaddam-Taaheri et al. 1985)

$$D_{\parallel}(v_{\parallel}, t) = \frac{\pi}{2} \int_{k_{\parallel} > 0} \frac{d^3 k}{(2\pi)^3} \left(\frac{k_{\parallel}}{k} \right)^2 \\ \times W_k(t) \left(\frac{k_{\perp} v_{\perp}}{\omega_{ce}} \right)^2 \delta(\omega_{ce} - k_{\parallel} v_{\parallel}).$$

EM waves in a magnetized plasma
(Karney 1986)

$$D_w = \sum_n \frac{q_e^2}{m_e^2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{V_p} \pi \delta[\omega(k) - k_{\parallel} v_{\parallel} - n\Omega_e] a_n^* a_n,$$

resonance conditions

harmonic

FMS waves – wave-particle interaction via moving
mirrors (Miller et al. 1996)

$$D_{\parallel\parallel} = (mc)^2 \Omega_H \frac{\pi}{8} \left(\frac{p_{\perp}}{mc} \right)^4 \frac{1}{\gamma^2} \left(\frac{c}{|v_{\parallel}|} \right)^3 \left(1 - \frac{v_A^2}{v_{\parallel}^2} \right) \left(\frac{v_A}{c} \right)^2 \frac{1}{U_B} \int_0^{\infty} dk \left(\frac{ck}{\Omega_H} \right) W_B(k)$$

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\partial}{\partial x_i} (v_i f_\sigma) - \frac{\partial}{\partial v_i} \left[\frac{F_i(\mathbf{x}, \mathbf{v})}{m} f_\sigma \right] - \frac{\partial}{\partial v_i} \left(\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle f_\sigma \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle f_\sigma \right)$$

$$\frac{1}{2} \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle = D_{ij} = D_{ij}^{(\text{col})} + D_{ij}^{(\text{turb})}$$

Collisional diffusion: from **particle-particle** interactions
 – drives $f_\sigma(\mathbf{x}, \mathbf{v})$ toward a Maxwellian (LTE)

Turbulent diffusion: from **wave-particle** interactions –
 drives $f_\sigma(\mathbf{x}, \mathbf{v})$ toward ???

$$D_{ij}^{(\text{col})}(\mathbf{v}) \sim \frac{4\pi e^4 \Lambda n' v_{rms}'^2}{m^2} \frac{\delta_{ij}}{v^3}$$

$$D_{ij}^{(\text{turb})}(\mathbf{v}) = 4\pi^2 \frac{e^2}{m^2} \varepsilon_{\text{turb}} \tau_{ij}(\mathbf{v})$$

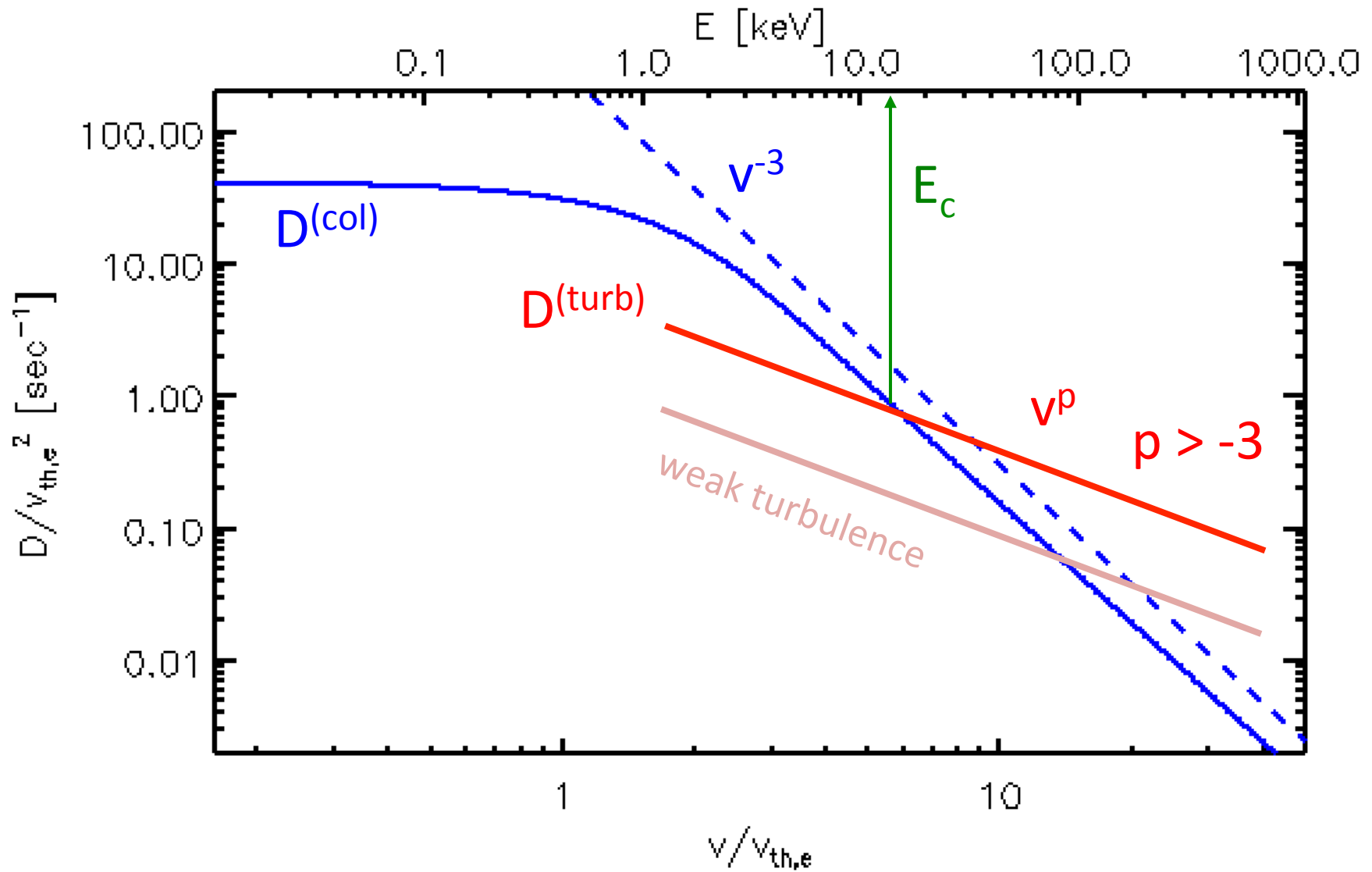
Q1: which term dominates?

in which part of phase space?

Q2: what form does $f_\sigma(\mathbf{x}, \mathbf{v})$ take

where $D^{(\text{turb})}$ dominates?

Q1: which term dominates?
in which part of phase space?



$$D_{ij}^{(\text{turb})}(\mathbf{v}) = 4\pi^2 \frac{e^2}{m^2} \varepsilon_{\text{turb}} \tau_{ij}(\mathbf{v}) \quad \text{say } p = -1 \quad (p > -3)$$

$$\tau_{ij}(\mathbf{v}) = \frac{1}{\bar{k} v}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{col}} = -\frac{\partial}{\partial v_i} \left\{ A_i(\mathbf{v}) \left[f - \frac{D_{ij}^{(\text{turb})}(\mathbf{v})}{A_i(\mathbf{v})} \frac{\partial f}{\partial v_j} \right] \right\}$$

Wave number
characterizing
spectrum

$$A_i(\mathbf{v}) = -\frac{4\pi e^4 \Lambda n v_i}{m^2 v^3} \quad \text{so} \quad \frac{D_{ij}^{(\text{turb})}(\mathbf{v})}{A_i(\mathbf{v})} = -\frac{\pi \varepsilon_{\text{turb}} v^2}{e^2 n \Lambda \bar{k} v_i} \delta_{ij}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{col}} = 0 \quad \Leftrightarrow \quad \underbrace{f(\mathbf{v}) \propto |\mathbf{v}|^{-\xi}}_{\text{power law spectrum}} \quad \text{w/}$$

$$\xi = \frac{e^2 n \Lambda \bar{k}}{\pi \varepsilon_{\text{turb}}}$$

power law
spectrum

See Bian et al. (2014)

Wave-particle
interaction w/

$$D_{ij}^{(\text{turb})}(\mathbf{v}) = 4\pi^2 \frac{e^2}{m^2} \frac{\varepsilon_{\text{turb}}}{\bar{k} v}$$

$$f(\mathbf{v}) \propto |\mathbf{v}|^{-\xi} \quad \xi = \frac{e^2 n \Lambda \bar{k}}{\pi \varepsilon_{\text{turb}}}$$

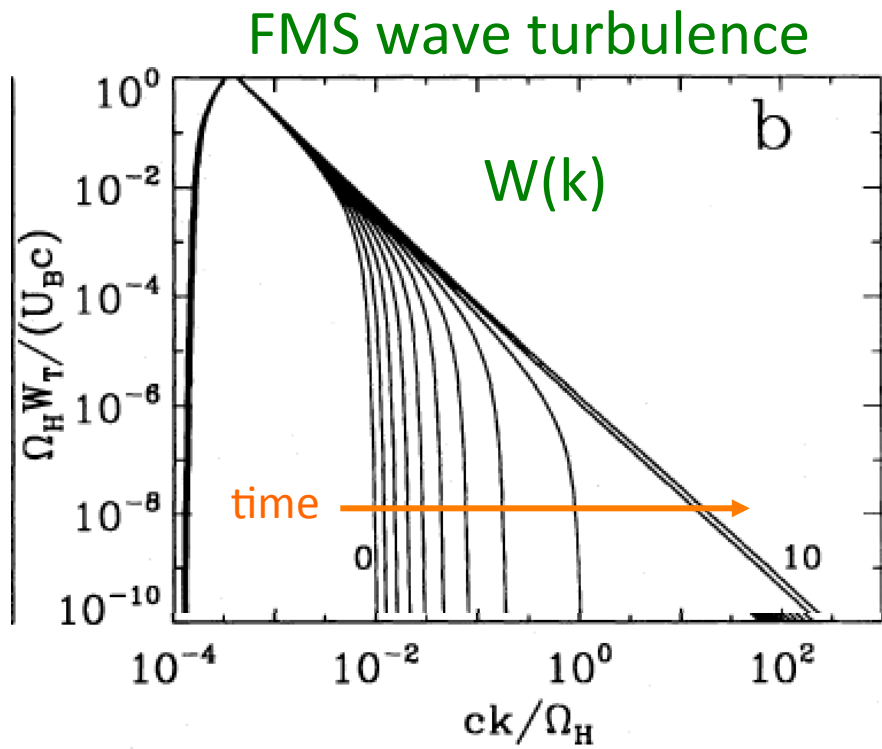
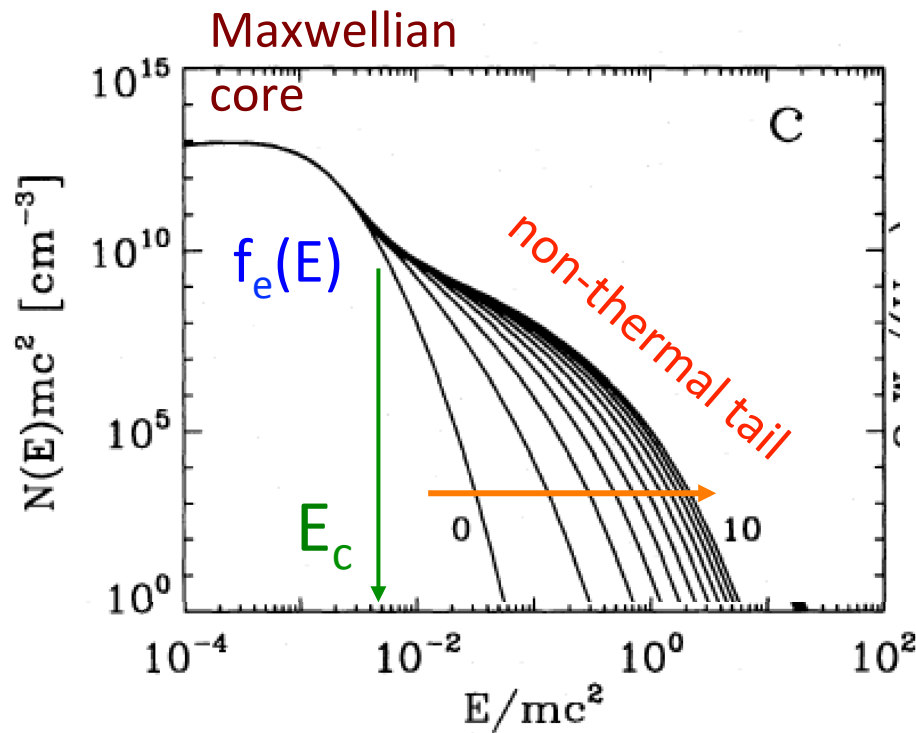
- Produces power law tail in distribution function
- Energy source: turbulence of waves
- Acceleration via wave-particle interaction
- more vigorous turbulence \rightarrow harder spectrum
(i.e. smaller ξ)

Stochastic acceleration in action

Miller *et al.* (1996)

FMS waves – wave-particle interaction via moving mirrors

$$D_{zz}^{(\text{turb})}(\mathbf{v}) = \frac{\pi}{8} \frac{\bar{k}}{m_p n} v \varepsilon_{\text{turb}}$$



Diffusive Shock acceleration

Assumes wave turbulence* producing **little/no energy change** [sic] but **very effective pitch angle scattering**

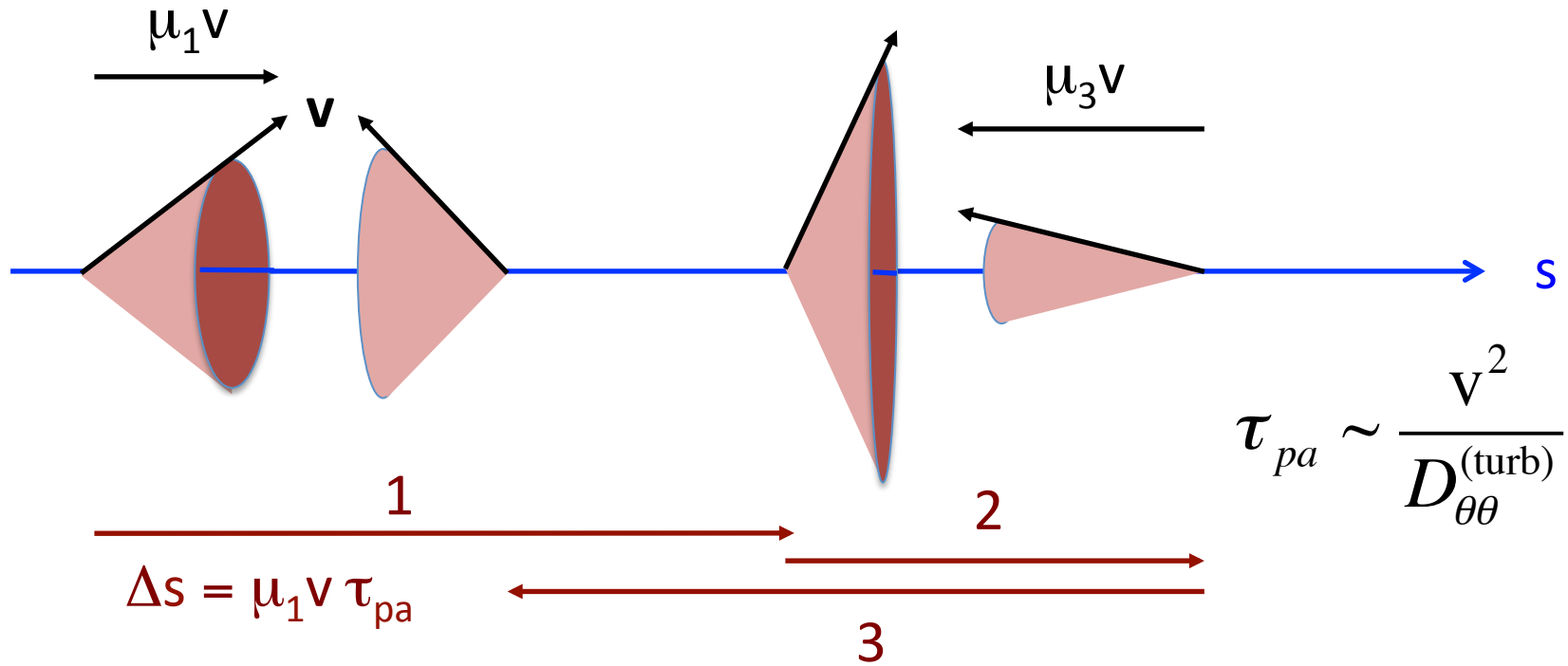
$$\left(\frac{\partial \tilde{f}(v, \mu)}{\partial t} \right)_{\text{col}} = \frac{\partial}{\partial v} \left[\cancel{D_{vv}^{(\text{turb})}} \frac{\partial \tilde{f}}{\partial v} \right] + \underbrace{\frac{\partial}{\partial \mu} \left[\frac{D_{\theta\theta}^{(\text{turb})}}{v^2} (1 - \mu^2) \frac{\partial \tilde{f}}{\partial \mu} \right]}_{\text{dominant term}}$$

$$\tau_{pa} \sim \frac{v^2}{D_{\theta\theta}^{(\text{turb})}} \quad \text{is v. small} \rightarrow \frac{\partial \tilde{f}}{\partial \mu} \approx 0 \quad \text{isotopic distribution of velocity}$$

“little/no energy change” – yet still the waves can lead to particle acceleration – **given a shock**

* Alfvén wave turbulence has this property

Fact 1: Strong pitch angle scattering leads to a random walk along field line \rightarrow **spatial diffusion**



Random walk \rightarrow
spatial diffusion
w/ coefficient

$$\kappa \sim \langle \mu v \Delta s \rangle \sim v^2 \langle \mu^2 \tau_{pa} \rangle \sim v^4 \int_{-1}^1 \frac{\mu^2}{D_{\theta\theta}^{(turb)}} d\mu$$

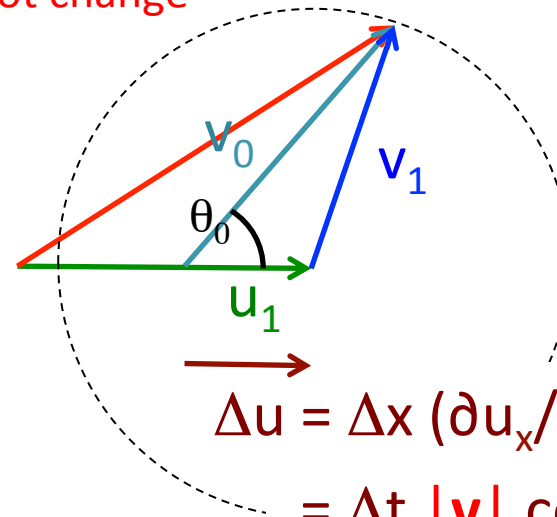
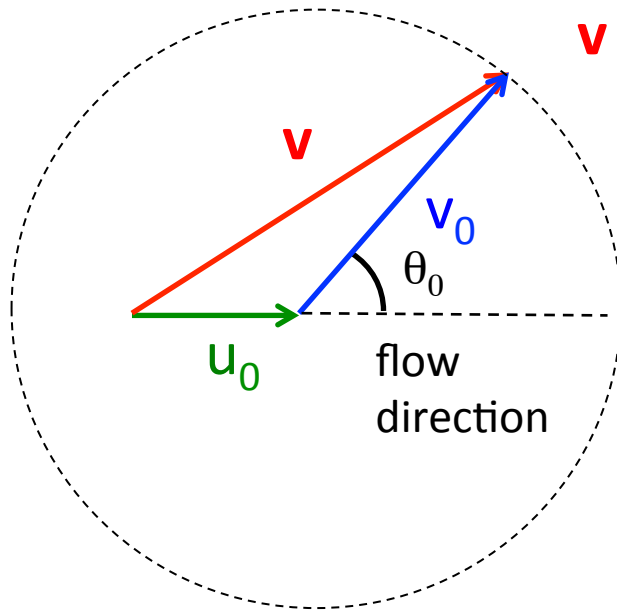
isotropic
distribution: $\frac{\partial \tilde{f}}{\partial \mu} \approx 0$ **BUT** pitch angle depends on
reference frame... in which
frame is it isotropic?

A: It is isotropic in reference frame of the **fluid** in which
waves are propagating – **frame moving at fluid velocity \mathbf{u}**

Fact 2: Strong pitch angle scattering leads to a
change in energy where $\nabla \cdot \mathbf{u} \neq 0$

Consider a place where $\nabla \cdot \mathbf{u} > 0$ & define x as the
gradient direction: $\nabla \cdot \mathbf{u} = \partial u_x / \partial x > 0$

A particle has velocity \mathbf{v} which does not change as it
moves up gradient of u_x . What happens to $|\mathbf{v} - \mathbf{u}|$?



$$\Delta u = \Delta x \left(\frac{\partial u_x}{\partial x} \right)$$

$$= \Delta t |\mathbf{v}| \cos(\theta_0) \left(\frac{\partial u_x}{\partial x} \right)$$

$$v_1 = \sqrt{v_0^2 + (\Delta u)^2 - 2v_0 \Delta u \cos(\theta_0)} \approx v_0 - \Delta u \cos(\theta_0) \quad \text{assume } u \ll |\mathbf{v}|$$

$$\Delta v = v_1 - v_0 = -\Delta u \cos(\theta_0) = -\Delta t |\mathbf{v}| \cos^2(\theta_0) \frac{\partial u_x}{\partial x}$$

$$\left\langle \frac{\Delta v}{\Delta t} \right\rangle_{\theta_0} = -\frac{1}{3} |\mathbf{v}| \frac{\partial u_x}{\partial x} = -\frac{1}{3} |\mathbf{v}| (\nabla \cdot \mathbf{u}) \quad \text{analogous to } A(\mathbf{v}) \text{ in F-P}$$

particles **gain** energy crossing shock ($\nabla \cdot \mathbf{u} < 0$)

Assuming dominant pitch-angle scattering

$$\underbrace{\frac{\partial \tilde{f}(v, \mu)}{\partial t} + \mu v \frac{\partial \tilde{f}}{\partial x}}_{\text{Green}} = \frac{\partial}{\partial \mu} \left[\underbrace{\frac{D_{\theta\theta}^{(\text{turb})}}{v^2} (1 - \mu^2) \frac{\partial \tilde{f}}{\partial \mu}}_{\text{Blue}} \right]$$

Fokker-Planck



$$\underbrace{\frac{\partial f(x, v)}{\partial t} + u_x \frac{\partial f}{\partial x}}_{\text{Green}} = \frac{1}{3} \left(\frac{\partial u_x}{\partial x} \right) v \frac{\partial f}{\partial v} + \frac{\partial}{\partial x} \left(\underbrace{\kappa \frac{\partial f}{\partial x}}_{\text{Blue}} \right)$$

Parker cosmic ray Eqn.

For rigorous derivations see Gleeson & Axford (1967), Skilling (1975), or Forbes & Priest (2000) 13.3

$$\kappa = \frac{v^4}{8} \int_{-1}^1 \frac{(1 - \mu)^2}{D_{\theta\theta}^{(\text{turb})}} d\mu$$

steady

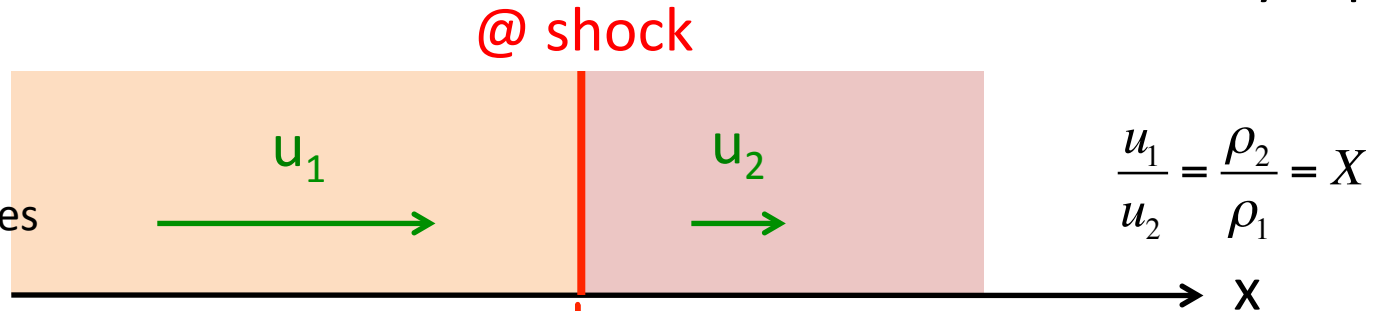
$$\frac{\partial f(x, v)}{\partial t} + u_x \frac{\partial f}{\partial x} = \frac{1}{3} \left(\frac{\partial u_x}{\partial x} \right) v \frac{\partial f}{\partial v} + \frac{\partial}{\partial x} \left(\kappa \frac{\partial f}{\partial x} \right)$$

Parker
cosmic
ray Eqn.

$$f \rightarrow 0$$

$$x \rightarrow -\infty$$

no NT particles
upstream



@ shock

$$\frac{\partial}{\partial x} \left(u_1 f - \kappa \frac{\partial f}{\partial x} \right) = 0$$

$$\left[\left[\kappa \frac{\partial f}{\partial x} \right] \right] = \frac{1}{3} (u_1 - u_2) v \frac{\partial f}{\partial v} \Big|_{x=0}$$

$$\frac{\partial}{\partial x} \left(u_2 f - \kappa \frac{\partial f}{\partial x} \right) = 0$$

$$f(x, v) = B(v) \exp\left(\frac{u_1}{\kappa} x\right)$$

$$[[f]] = 0$$

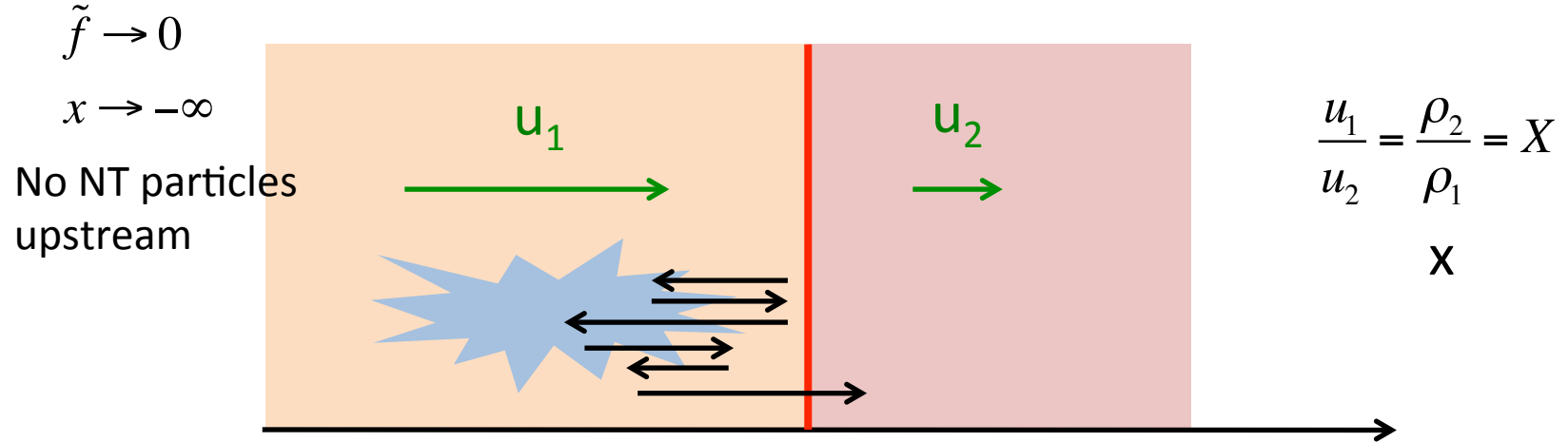
$$f(x, v) = B(v)$$

$$\left[\left[\kappa \frac{\partial f}{\partial x} \right] \right] = -u_1 B(v) = \frac{1}{3} (u_1 - u_2) v B'(v)$$

power
law

$$B(v) = v^{-\xi}$$

$$\xi = \frac{3u_1}{u_1 - u_2} = \frac{3X}{X - 1}$$



turbulence scatters
 particles back to shock
 – shock energized
 particles

$$f(x, v) = f_0 v^{-\xi}, \quad x > 0$$

$$\xi = \frac{3X}{X-1}$$

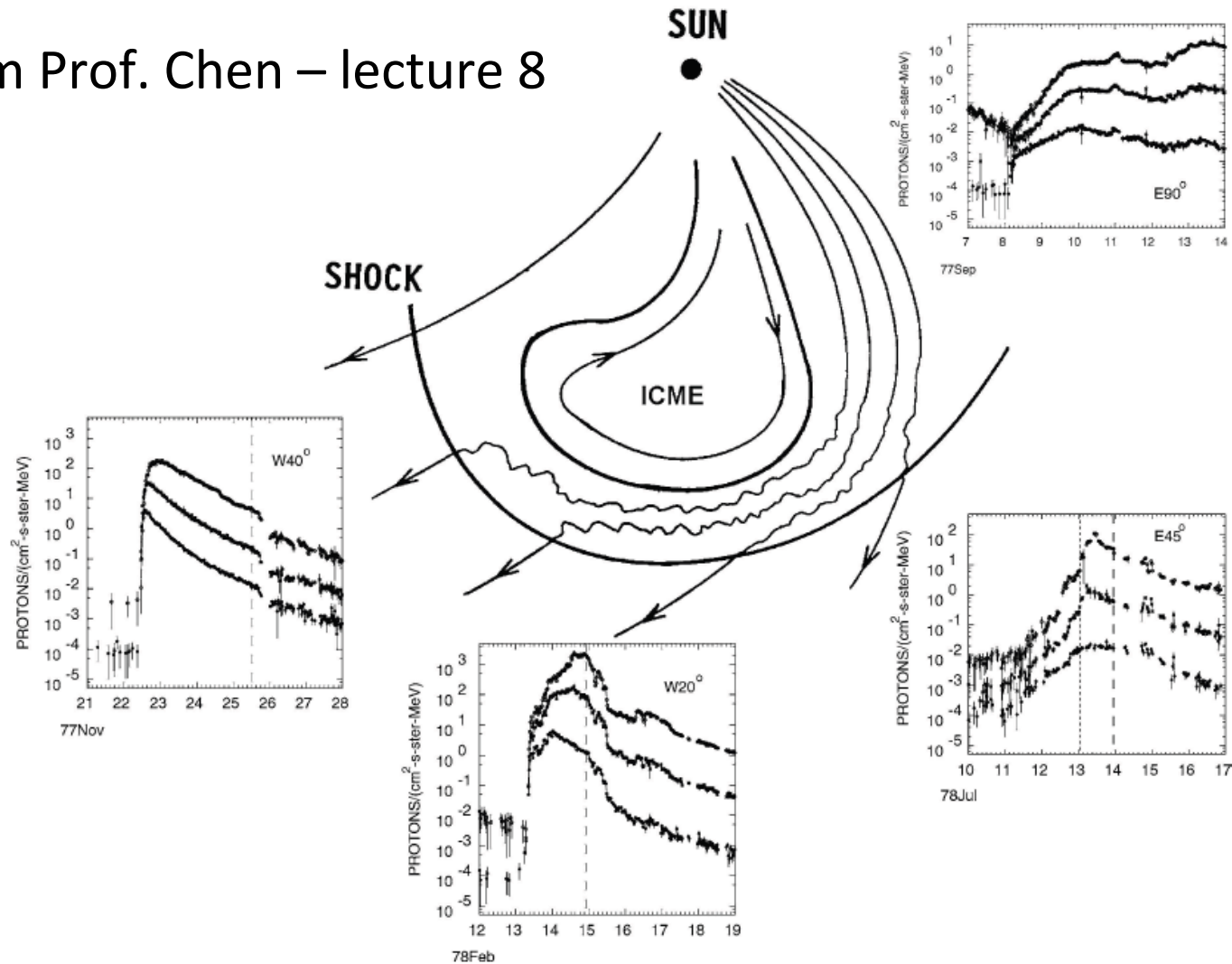
MHD shocks: $1 < X < 4$

$\rightarrow \infty > \xi > 4$

tends to be more effective w/ ions

Shocks are good particle accelerators

From Prof. Chen – lecture 8



From Cane & Lario 2006

Summary

- F-P eqn. can include effect of random plasma waves on particles – wave-particle interactions
- Effect captured in $D_{ij}^{(\text{turb})}$
- Can lead to non-thermal tail – incl. power law
- Wave-particle pitch-angle scattering + shock → diffusive shock acceleration
- Viable explanations for NT tail – is it correct?
How could we tell?

State of the field today

- SEP ion acceleration – diffusive shock (general consensus IMHO)
- Flare acceleration (e^- or ion) – no consensus IMHO
 - SA models popular
 - need to account for observations of $\delta(t)$, $E_c(t)$ etc.
 - SA models not coupled to global flare model – no self-consistent energy supply (start w/ given turbulence)