Flare models include some version of the radiative loss function $\Lambda(T)$. A monomial approximation

$$\Lambda(T) \simeq \Lambda_0 T_6^\alpha, \quad \Lambda_0 = 1.2 \times 10^{-22} \text{erg cm}^{-3} \text{s}^{-1} , \quad (1)$$

permits an analytic treatment. Here $T_6$ is the temperature expressed in units of MegaKelsvins, and $\alpha$ is some power-law index. Lecture 9 uses this form with $\alpha = -1/2$, which we argue to be a reasonable approximation to the actual function. Zero-dimensional models must use a version of this function, but it represents the average loss as a function of the average temperature. Once again a monomial approximation permits analytic treatment, but it is not clear that $\alpha = -1/2$ is still appropriate: the average of a function is not the same as the function of the average. We therefore re-derive some of our initial results, but using a general choice of $\alpha$.

a. Using the form (1) for the radiative loss function, find an expression for the radiative cooling time, $\tau_{\text{rad}}$. Express this in terms of $T_6$ and $n_{10}$, the electron density in units of $10^{10} \text{ cm}^{-3}$.

b. Use these results of part a. to find the relation between $T_6$ and $n_{10}$ for which the radiative and conductive times are equal: $\tau_{\text{rad}} = \tau_{\text{cond}}$. Write $n_{10}$ explicitly in terms of $T_6$ and $L_9$, the full length of the loop in units of $10^9 \text{ cm}$. This relations also approximates the condition for mechanical equilibrium. For which $\alpha$ are $T$ and $n$ correlated with one another, rather than anti-correlated?

c. Under the assumption of Antiochos & Sturrock (1978), evaporation decreases $T$ and increases $n$, keeping constant their product $n_e T$, until $\tau_{\text{rad}} = \tau_{\text{cond}}$, at which point evaporation ends. The constant along which evaporation evolves is set by the total energy (per unit area) released in the flare, $\mathcal{E}$ (erg cm$^{-2}$)

$$n_e T = \frac{\mathcal{E}}{3L k_b} . \quad (2)$$

Use this to find the peak density, $n_*$, achieved at the end of evaporation, along with the corresponding temperature, $T_*$. Write down expressions for $n_{10*}$ and $T_{6*}$, involving only $\mathcal{E}_9$, $L_9$ and $\alpha$.

d. Next assume that the radiative cooling process occurs in quasi-static equilibrium, so $\tau_{\text{rad}} = \tau_{\text{cond}}$, as found in part b. Klimchuk et al (2008) set the enthalpy equal to the difference between the conductive flux and the radiative loss form the transition region. They further set that loss to be 4 times the coronal radiative loss yields an energy equation

$$\frac{3 dp}{2 dt} = -5 n_e^2 \Lambda(T) , \quad (3)$$

where $p = 2 n_e k_b T$. Solve this to obtain an expression for $T_6(t)$, resembling slide 29 of lecture 9, but involving an arbitrary index $\alpha$ rather than the choice $\alpha = -1/2$. The solution should take the form

$$T(t) = T_* \left(1 + \omega t\right)^{-\mu} , \quad (4)$$

\textsuperscript{1}The result of the assumptions is that conductive and enthalpy fluxes move energy around without changing it. All changes come from radiative losses which amount to $4 + 1 = 5$ times the coronal loss alone.
<table>
<thead>
<tr>
<th>ion</th>
<th>$\lambda$ (Å)</th>
<th>$G^{(0)}_{\lambda}$ (10^{-25} \text{erg cm}^3 \text{s}^{-1} \text{sr}^{-1})$</th>
<th>$T_{\lambda}$ (MK)</th>
<th>$\sigma_{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fe XXI</td>
<td>128.7</td>
<td>1.64</td>
<td>11.51</td>
<td>0.30</td>
</tr>
<tr>
<td>Fe XVIII</td>
<td>93.9</td>
<td>1.43</td>
<td>6.91</td>
<td>0.44</td>
</tr>
<tr>
<td>Fe XIV</td>
<td>211.3</td>
<td>6.13</td>
<td>1.99</td>
<td>0.25</td>
</tr>
<tr>
<td>Fe IX</td>
<td>171.1</td>
<td>37.84</td>
<td>0.82</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 1: Parameters defining the function $G_{\lambda}(T)$ for spectral lines in 4 different ions.

for constants of $\mu$ and $\omega$ you can write as explicit functions of $E_0, L_0$ and $\alpha$. Here we have taken $t = 0$ to be the beginning of radiative cooling — the time of peak density. Are there any values of $\alpha$ for which the solution vanishes in finite time?

e. Contribution functions from most optically thin spectral lines appear parabolic on log-log plots (see slides from lecture 11). They may therefore be approximated analytically as

$$G_{\lambda}(T) = G^{(0)}_{\lambda} \exp \left[ -\frac{\ln^2(T/T_{\lambda})}{\sigma^2_{\lambda}} \right],$$  \hspace{1cm} (5)

where the constants $G^{(0)}_{\lambda}, T_{\lambda}$ and $\sigma_{\lambda}$ are constants for that particular spectral line. Values for 4 different spectral lines are given in table 1. Adopting this parameterization, show that during the radiative cooling phase the emissivity of a line, $\varepsilon_{\lambda}(t)$, will peak at a temperature

$$T_{\lambda, \text{pk}} = \beta T_{\lambda},$$  \hspace{1cm} (6)

for a factor $\beta$ dependent only on $\alpha$ and $\sigma_{\lambda}$. Is $\beta$ greater than or less than unity? Why? [The easiest approach to this result is to express $\ln(\varepsilon_{\lambda})$ as a polynomial in $\ln(T)$, and find the maximum of that.]

f. As the loop cools it passes through peak emission at some time $t_{\lambda}$. The lifetime of its emission in this spectral line, $\Delta \tau_{\lambda}$, can be analytically defined though the relation

$$\frac{1}{\varepsilon_{\lambda}} \left. \frac{d^2 \varepsilon_{\lambda}}{dt^2} \right|_{t_{\lambda}} = -\frac{2}{\Delta \tau_{\lambda}^2}.$$  \hspace{1cm} (7)

Use the results above to show that

$$\Delta \tau_{\lambda} = \frac{\sigma_{\lambda}}{\omega \mu} \left( \frac{T_{\lambda, \text{pk}}}{T^*} \right)^{-\nu},$$  \hspace{1cm} (8)

for some index $\nu$, depending on $\alpha$, and $\omega$ and $\mu$ defined through eq. (4).

g. Now consider a particular flare with $L = 5 \times 10^9 \text{cm}$ and $E = 2 \times 10^{12} \text{erg cm}^{-2}$ (the same values used to produce slide 34 of lecture 9). Return to the conventional choice of radiative loss function by setting $\alpha = -1/2$ (consistent with EBTEL). Consider each of the spectral lines in table 1 for which the loop cools through the peak during its radiative phase. For each of them find the time and duration of its peak emission, $t_{\lambda}$ and $\Delta \tau_{\lambda}$. 

2
Solutions:
a. The radiative cooling time is
\[
\tau_{\text{rad}} = \frac{3}{2} \frac{n_e k_b T}{n_c^2 \Lambda(T)} = \frac{3}{2} \frac{3 \times 1.38 n_{10} T_6}{n_c^2 \Lambda(T)} = \frac{3 \times 1.38 n_{10} T_6}{\Lambda_0 10^{20} n_{10}^2 T_6^\alpha} = (345 \, \text{s}) T_6^{1-\alpha} n_{10}^{-1} \ . \quad (9)
\]
b. The conductive cooling time is still the same as in the lecture notes,
\[
\tau_{\text{cond}} = \frac{3}{8} \frac{n_e k_b T}{\kappa T^{7/2}/T^2} = \frac{3 \times 1.38 n_{10} T_6}{1.14 \times 10^{-3} T_6^{7/2}/L_g^2} = (3622 \, \text{s}) T_6^{-5/2} n_{10}^2 L_g^{-2} \ . \quad (10)
\]
Equating these yields the relation
\[
\frac{\tau_{\text{rad}}}{\tau_{\text{cond}}} = \frac{345}{3622} T_6^{7/2-\alpha} n_{10}^{-2} L_g^{-2} = 1 \ , \quad (11)
\]
or
\[
n_{10} = 0.31 T_6^{(7-2\alpha)/4} L_g^{-1} \ . \quad (12)
\]
Temperature and density are correlated only for \( \alpha < 7/2 \). Otherwise they are anti-correlated. Such anti-correlation is a signature of a radiative instability.
c. The evaporation must occur along the curve
\[
n_{10} T_6 = 10^{-16} \frac{E}{3 k_b L} = 0.24 \mathcal{E}_9 L_g^{-1} \ . \quad (13)
\]
Evaporation continues until \( \tau_{\text{rad}} = \tau_{\text{cond}} \), which occurs when eq. (12) is satisfied. Introducing that yields
\[
T_6^{(11-2\alpha)/4} = 0.77 \mathcal{E}_9 \quad \implies \quad T_{6,*} = (0.77 \mathcal{E}_9)^{4/(11-2\alpha)} \ . \quad (14)
\]
Using this in eq. (12) yields the peak density
\[
n_{10,*} = 0.31 L_g^{-1} (0.77 \mathcal{E}_9)^{(7-2\alpha)/(11-2\alpha)} \ . \quad (15)
\]
As a check on these results we set \( \alpha = -1/2 \) to obtain
\[
T_{6,*} = (0.77 \mathcal{E}_9)^{1/3} = 0.93 \mathcal{E}_9^{1/3} \ , \quad n_{10,*} = 0.27 L_g^{-1} \mathcal{E}_9^{2/3} \ .
\]
These expression agree with those on slide 32 of lecture 9, after using the fact that \( \mathcal{E} = EL/V \).
d. Dividing the energy equation, eq. (3), by \( p \) yields
\[
\frac{3}{2} \frac{dp}{p} \frac{dt}{dt} = - \frac{5}{2} \frac{n_c^2 p}{p} \Lambda(T) = - \frac{3}{2} \frac{5}{2} \frac{\tau_{\text{rad}}}{\tau_{\text{cond}}} \ , \quad (16)
\]
after introducing the cooling times which are equal. (Trick-of-the-trade: it is frequently easier to formulate ODEs in terms of logarithmic derivatives, as I am doing here. This cuts down on the number of constants needed.) If we now substitute \( p = 2n_e k_b T \) we obtain

\[
\frac{1}{p} \frac{dp}{dt} = \frac{1}{T} \frac{dT}{dt} + \frac{1}{n_e} \frac{dn_e}{dt} = - \frac{5}{\tau_{\text{cond}}} = -(1.4 \times 10^{-3} \text{s}^{-1}) T_6^{5/2} n_1^{-1} L_9^{-2} . \quad (17)
\]

We now use eq. (12) to re-write the logarithmic derivative of density

\[
\frac{1}{n_e} \frac{dn_e}{dt} = \frac{1}{n_{10}} \frac{dn_{10}}{dt} = \frac{d \ln(n_{10})}{dt} = \frac{7-2\alpha}{4} \frac{d \ln(T_6)}{dt} = \frac{7-2\alpha}{4} \frac{1}{T} \frac{dT}{dt} . \quad (18)
\]

Placing this into eq. (17) yields a simple equation for \( T_6 \) alone

\[
\frac{1}{p} \frac{dp}{dt} = \left( 1 + \frac{7-2\alpha}{4} \right) \frac{1}{T} \frac{dT}{dt} = \left( \frac{11-2\alpha}{4} \right) \frac{1}{T_6} \frac{dT_6}{dt} = -(1.4 \times 10^{-3} \text{s}^{-1}) T_6^{5/2} \left[ 0.31 T_6^{(7-2\alpha)/4} L_9^{-1} \right]^{-1} L_9^{-2} = -(4.5 \times 10^{-3} \text{s}^{-1}) T_6^{(3+2\alpha)/4} L_9^{-1} . \quad (19)
\]

This provides us with a simple ODE for \( T_6 \),

\[
\frac{dT_6}{dt} = -(4.5 \times 10^{-3} \text{s}^{-1}) \frac{4}{11-2\alpha} L_9^{-1} T_6^{1+(3+2\alpha)/4} . \quad (20)
\]

Dividing both sides by \( T_6^{1+(3+2\alpha)/4} \) yields an equation

\[
T_6^{-1-(3+2\alpha)/4} \frac{dT_6}{dt} = \frac{4}{3+2\alpha} \frac{dT_6^{-1-(3+2\alpha)/4}}{dt} = -(4.5 \times 10^{-3} \text{s}^{-1}) \frac{4}{11-2\alpha} L_9^{-1} ,
\]

which can be re-cast in the form

\[
\frac{d}{dt} \left[ T_6^{-(3+2\alpha)/4} \right] = \frac{3+2\alpha}{11-2\alpha} \frac{4}{(4.5 \times 10^{-3} \text{s}^{-1})} \frac{1}{L_9} , \quad (21)
\]

— very easy to solve. Dividing both sides by \( T_{6,*}^{-(3+2\alpha)/4} \) yields an equation

\[
\frac{d}{dt} \left( \frac{T_6}{T_{6,*}} \right)^{-(3+2\alpha)/4} = (4.5 \times 10^{-3} \text{s}^{-1}) \frac{3+2\alpha}{11-2\alpha} \frac{T_{6,*}^{(3+2\alpha)/4}}{L_9} = \omega , \quad (22)
\]

where we have defined the right hand side to be \( \omega \), in anticipation of the form in (4). Using eq. (14) yields an explicit version

\[
\omega = \frac{1}{224 \text{s}} \frac{3+2\alpha}{11-2\alpha} \frac{(0.77 \mathcal{E}_9)^{(3+2\alpha)/(11-2\alpha)}}{L_9} , \quad (23)
\]

depending only on \( \mathcal{E}_9 \) and \( L_9 \), as well as on \( \alpha \).

Defining \( t = 0 \) to be the time at which radiative cooling begins, and thus \( T_6 = T_{6,*} \), we have the solution

\[
\frac{dT_6}{dt} = -(4.5 \times 10^{-3} \text{s}^{-1}) \frac{4}{11-2\alpha} L_9^{-1} T_6^{1+(3+2\alpha)/4} .
\]
\[
T(t) = T_\star \left(1 + \omega t\right)^{-\mu}, \quad \mu = \frac{4}{3 + 2\alpha}.
\] (24)

We can check this using the limit \(\alpha = -1/2\), for which \(\mu = 2\) and we find
\[
T(t) = T_\star \left(1 + \omega t\right)^{-2},
\]

Since the loop is cooling, expression (24) must be a monotonically decreasing function of time. In the case
\[
\alpha < -\frac{3}{2},
\] (25)
we find the exponent \(\mu < 0\). Expression (23) reveals that for this same case, \(\omega < 0\). Thus the temperature drops to zero at \(t = 1/|\omega|\).

e. The emissivity of spectral line \(\lambda\) is
\[
\varepsilon_\lambda = 4\pi n_e^2 G_\lambda(T) = 4\pi n_e^2 G_\lambda^{(0)} \exp \left[ -\frac{\ln^2(T/T_\lambda)}{\sigma^2} \right].
\] (26)

Taking its natural log and using eq. (12) to eliminate \(n_e\) yields
\[
\ln(\varepsilon_\lambda) = \frac{7 - 2\alpha}{2} \ln(T) - \frac{1}{\sigma^2} \left[ \ln(T) - \ln(T_\lambda) \right]^2 + \text{const.} = f(T),
\] (27)
where the constants are not important for obtaining a maximum. Taking the derivative of this expression w.r.t. \(T\), and setting it to zero, yields the equation
\[
T f'(T) = \frac{7 - 2\alpha}{2} - \frac{2}{\sigma^2} \left[ \ln(T) - \ln(T_\lambda) \right] = 0.
\] (28)
The temperature at peak emissivity is
\[
T_{\lambda,\text{pk}} = T_\lambda \exp \left[ \frac{\ln^2(T/T_\lambda)}{\sigma^2} \right] = \beta T_\lambda,
\] (29)
Provided temperature and density both decrease during radiative cooling (i.e. \(\alpha < 7/2\)) the coefficient \(\beta > 1\), and the emissivity will peak at a temperature slightly higher than the contribution function does. This occurs as the density is dropping just fast enough to offset the increase in contribution function as \(T\) decreases toward \(T_\lambda\).

f. The time-dependent emissivity can be written
\[
\varepsilon_\lambda = \exp \left\{ f[T(t)] \right\},
\] (30)
where \(f(T)\) is defined in eq. (27). Taking its first derivative yields
\[
\frac{d\varepsilon_\lambda}{dt} = \frac{dT}{dt} f'(T) \exp \left\{ f[T(t)] \right\} = \frac{dT}{dt} f'(T) \varepsilon_\lambda,
\] (31)
after two applications of the chain rule. Naturally this vanishes at $T = T_{\lambda,pk}$, since that is where $f'(T) = 0$. The only non-vanishing contribution to the second derivative will come from the term with $f''(T)$, since every other terms will involve at least one factor of $f'(T)$, which will vanish. The result is

$$
\left. \frac{d^2 \varepsilon_{\lambda}}{dt^2} \right|_{T_{\lambda,pk}} = \left( \frac{dT}{dt} \right)^2 f''(T_{\lambda,pk}) \varepsilon_{\lambda} .
$$

Two derivatives of (27) yields

$$
f''(T_{\lambda,pk}) = -\frac{2}{T_{\lambda,pk}^2 \sigma^2} .
$$

Next we take the time derivative of eq. (24) to obtain

$$
\frac{dT}{dt} = -\mu \omega T_*(1 + \omega t)^{-\mu-1} = -\mu \omega T_* \left( \frac{T}{T_*} \right)^{(\mu+1)/\mu} .
$$

Combing these pieces yields

$$
\frac{1}{\varepsilon_{\lambda}} \left. \frac{d^2 \varepsilon_{\lambda}}{dt^2} \right|_{T_{\lambda,pk}} = -\frac{2\mu^2\omega^2}{\sigma^2} \left( \frac{T_{\lambda,pk}}{T_*} \right)^{2/\mu} .
$$

equating with $2/\Delta \tau^2_{\lambda}$ yields the emission lifetime

$$
\Delta \tau_{\lambda} = \frac{\sigma}{\omega \mu} \left( \frac{T_{\lambda,pk}}{T_*} \right)^{-1/\mu} ,
$$

This matches eq. (8) for an index

$$
\frac{1}{\mu} = \frac{3 + 2\alpha}{4} .
$$
The given values correspond to $L_\theta = 5$ and $E_\theta = 2000$. Using the latter in eq. (14) yields

$$T_{6, \ast} = (0.77 \cdot 2000)^{1/3} = 11.5 \ .$$

From eq. (29) we find

$$\beta = e^{2\sigma^2} ,$$

which yields a different value, greater than one, for each spectral line according to its value of $\sigma$. These are listed in table 2. Multiplying these by the different values of $T_\lambda$ and then dividing by $T_\ast = 11.5$ MK yields the ratio $T_{\lambda, pk}/T_\ast$ also listed in the table. The first ratio, for Fe XXI, is greater than one, meaning the 128.7Å line will not peak during the radiative phase. According to the wording of the problem, we need not compute a time for this line. (If we tried we would find a negative cooling time — strange!).

For the remaining three spectral lines, the peak time is found by setting $T = T_{\lambda, pk}$ in eq. (24) and solving for time. The result is

$$t_\lambda = \frac{1}{\omega} \left[ \left( \frac{T_{\lambda, pk}}{T_\ast} \right)^{1/2} - 1 \right] .$$

Equation (23) yields

$$\frac{1}{\omega} = (224 s) \cdot \frac{12}{2} \cdot \frac{5}{(0.77 \cdot 2000)^{1/6}} = 1.9 \times 10^3 s \ .$$

This is the characteristic cooling time for this flare loop. It is reassuringly comparable to cooling times we often encounter: some fraction of an hour (here it is 31 min. 40 s).

Combining this values with eq. (40) yields the delays given in table 2. The lifetime is given by

$$\Delta \tau_\lambda = \frac{\sigma}{\omega \mu} \left( \frac{T_{\lambda, pk}}{T_\ast} \right)^{-2/\mu} = (1.0 \times 10^3 s) \sigma \left( \frac{T_{\lambda, pk}}{T_\ast} \right)^{-1} .$$

\begin{table}[h]
\centering
\begin{tabular}{llllll}
\hline
ion & $\lambda$ & $\beta$ & $T_{\lambda, pk}/T_\ast$ & $t_\lambda$ & $\Delta \tau_\lambda$ \\
Fe XXI & 128.7 & 1.20 & 1.20 & – & – \\
Fe XVIII & 93.9 & 1.47 & 0.88 & 134 & 463 \\
Fe XIV & 211.3 & 1.13 & 0.20 & 2,500 & 560 \\
Fe IX & 171.1 & 1.42 & 0.10 & 4,200 & 1,300 \\
\hline
\end{tabular}
\caption{Derived parameters for 4 different ions.}
\end{table}