

A ship's wake — DANA LONGCOPE 1/17/2009, REVISED 11/23/2015

The properties of a ship's wake follow entirely from the dispersion relation for deep-water waves. In still water, a small-amplitude disturbance the the surface with horizontal wave number $\mathbf{k} = k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}}$ oscillates at angular frequencies

$$\omega(\mathbf{k}) = \pm \sqrt{g|\mathbf{k}|} \quad , \quad (1)$$

where g is the gravitational acceleration, and the upper and lower signs give the two branches of the dispersion relation necessary to produce a surface displacement with real values. The form of this relation can be justified by dimensional analysis once it is recognized that the only relevant quantities are g and the magnitude of the wave-vector itself.

We now place ourselves in the reference frame of a ship moving over the still water at velocity \mathbf{u} . In this frame, the water appears to be moving at velocity $-\mathbf{u}$, and the dispersion relations are doppler shifted to

$$\omega_{\pm}(\mathbf{k}) = \pm \sqrt{g|\mathbf{k}|} - \mathbf{u} \cdot \mathbf{k} \quad . \quad (2)$$

We learn a great deal about the problem by simply plotting the dispersion relation, in (k_x, k_y) space, as done in fig. 1 after taking $\mathbf{u} = u\hat{\mathbf{x}}$. From this we see that frequency is positive, $\omega_+ > 0$, for all wave vectors $k_x < 0$. There is a small region where contours enclose the origin $\mathbf{k} = 0$; this occurs for $0 < \omega < g/4u^2$, shown by the magenta contour. For frequencies outside this range, as well as for a second branch within it, contours run roughly vertically in (k_x, k_y) space, with a leftward bulge near $k_y = 0$.

The group velocity of the ω_+ branch

$$\mathbf{v}_g = \frac{\partial \omega_+}{\partial \mathbf{k}} = \frac{1}{2} \sqrt{g|\mathbf{k}|} \frac{\mathbf{k}}{|\mathbf{k}|^2} - \mathbf{u} \quad (3)$$

is perpendicular to the contour shown in fig. 1 and is directed toward larger frequencies. For the roughly vertical contours the group velocity is leftward — it is directed downstream. Only for the wave vectors in the neighborhood of the origin for which $0 < \omega_+ < g/4u^2$, are up-stream group velocities possible.

From expression (2) it can be seen that the other branch of the dispersion relation has the property $\omega_-(\mathbf{k}) = -\omega_+(-\mathbf{k})$. This branch is necessary to accommodate a real solution. To any complex wave, $\sim e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_+(\mathbf{k})t}$, we must be able to add the complex conjugate

$$\left[e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_+(\mathbf{k})t} \right]^* = e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_+(\mathbf{k})t} = e^{i(-\mathbf{k})\cdot\mathbf{x} + i\omega_+(\mathbf{k})t} = e^{i(-\mathbf{k})\cdot\mathbf{x} - i\omega_-(-\mathbf{k})t} \quad .$$

Since this is a wave satisfying the dispersion relation of the other branch, and with the opposite wave vector, it is also a solution of the linearized equation and can therefore be added to yield the real function we seek. Given that $\omega_+(\mathbf{k}) = -\omega_-(-\mathbf{k})$ we can show that the group velocities are the same at corresponding wave vectors

$$\left. \frac{\partial \omega_+}{\partial \mathbf{k}} \right|_{\mathbf{k}} = \left. \frac{\partial \omega_-}{\partial \mathbf{k}} \right|_{-\mathbf{k}} \quad . \quad (4)$$

The wake from a boat is stationary in the co-moving frame. Setting $\omega_+ = 0$ in eq. (2) provides an expression for the blue curve in fig. 1

$$|\mathbf{k}| = \frac{g}{(\mathbf{u} \cdot \hat{\mathbf{k}})^2} \quad (5)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ is the direction of the wave vector. Placing this into eq. (3) gives the group velocity of the stationary waves

$$\mathbf{v}_g(\hat{\mathbf{k}}) = \frac{1}{2} \hat{\mathbf{k}} (\hat{\mathbf{k}} \cdot \mathbf{u}) - \mathbf{u} \quad , \quad (6)$$

which are shown as violet arrows on the right of fig. 1. It is evident in the figure that these all point approximately down stream, i.e. toward $-\mathbf{u}$. When $\hat{\mathbf{k}} = \hat{\mathbf{x}}$ the group velocity points exactly in this direction: $\mathbf{v}_g \parallel -\hat{\mathbf{x}}$. Then as θ_k increases or decreases $\mathbf{v}_g(\hat{\mathbf{k}})$ makes an angle to the downstream direction. Finally, as $\theta_k \rightarrow \pm\pi/2$, i.e. $\hat{\mathbf{k}} \rightarrow \pm\pi/2$, and thus $\hat{\mathbf{k}} \cdot \mathbf{u} \rightarrow 0$, the group velocity goes to $\mathbf{v}_g \rightarrow -\mathbf{u}$ and it

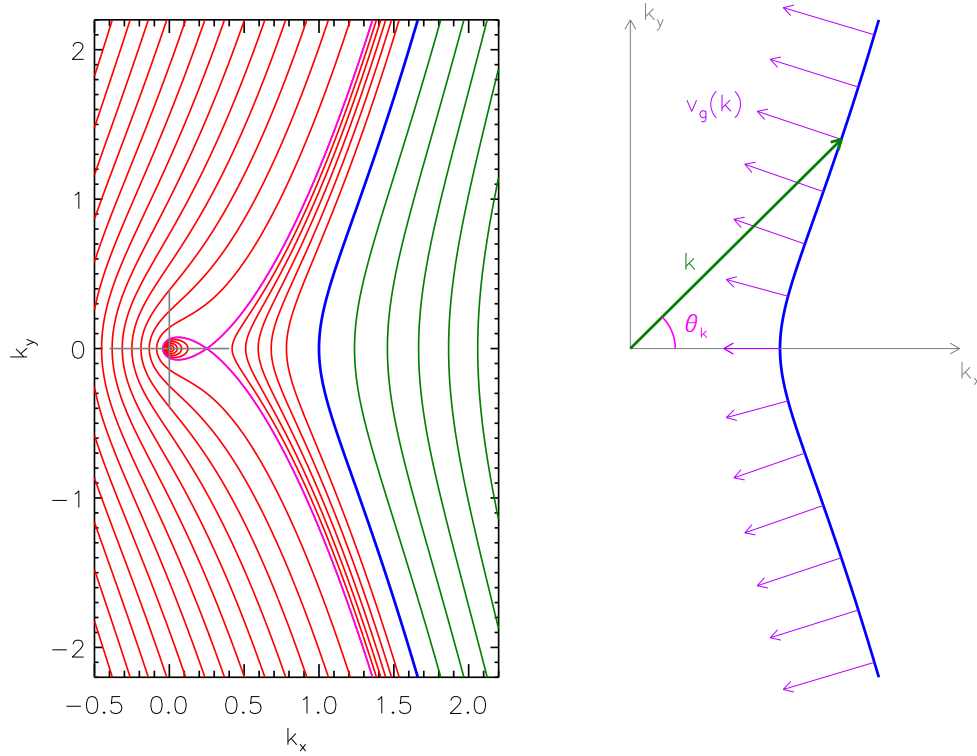


Figure 1: Contours of the dispersion relation $\omega_+(k_x, k_y)$, given by the upper sign in eq. (2) for $\mathbf{u} = u\hat{\mathbf{x}}$. Left shows contours of positive (negative) values in red (green) against axes whose gave vectors are scaled to g/u^2 . The curve $\omega = 0$ is shown as a thick blue contour, and repeated in isolation on the right. The magenta contour is the particular value $\omega = g/4u$. A green arrow on the right version shows a particular wave vector \mathbf{k} , making an angle θ_k . Violet arrows show the group velocity, \mathbf{v}_g at a series of representative points.

points directly downstream again. It thus seems that $\mathbf{v}_g(\hat{\mathbf{k}})$ sweeps over a range of downstream directions as $\hat{\mathbf{k}}$ goes from $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}} \rightarrow -\hat{\mathbf{y}}$. This limited range of angles defines a wedge we know as the *ship's wake*.

We have already evaluated the group velocities for wave vectors in three different directions

$$\mathbf{v}_g(\hat{\mathbf{y}}) = -\mathbf{u} = -u\hat{\mathbf{x}} \quad , \quad \mathbf{v}_g(\hat{\mathbf{x}}) = -\frac{1}{2}u\hat{\mathbf{x}} \quad , \quad \mathbf{v}_g(-\hat{\mathbf{y}}) = -u\hat{\mathbf{x}} \quad . \quad (7)$$

To these we add the two at $\theta_k = \pm\pi/4$,

$$\mathbf{v}_g\left(\frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{y}}\right) = u\left(-\frac{3}{4}\hat{\mathbf{x}} + \frac{1}{4}\hat{\mathbf{y}}\right) \quad , \quad \mathbf{v}_g\left(\frac{1}{\sqrt{2}}\hat{\mathbf{x}} - \frac{1}{\sqrt{2}}\hat{\mathbf{y}}\right) = u\left(-\frac{3}{4}\hat{\mathbf{x}} - \frac{1}{4}\hat{\mathbf{y}}\right) \quad . \quad (8)$$

Our five different wave-vector directions lead to group velocities arrayed in a square centered at $-(3/4)\mathbf{u}$. Motivated by this observation we re-arrange eq. (6) into the form

$$\mathbf{v}_g(\hat{\mathbf{k}}) = \frac{1}{4} \underbrace{[2\hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{u}) - \mathbf{u}]}_{u\hat{\mathbf{e}}(\hat{\mathbf{k}})} - \frac{3}{4}\mathbf{u} \quad , \quad (9)$$

leading to a new vector $\hat{\mathbf{e}}(\hat{\mathbf{k}})$. It can be shown that the factor in square brackets has magnitude $u = |\mathbf{u}|$ regardless of $\hat{\mathbf{k}}$, and thus that $\hat{\mathbf{e}}$ is a unit vector. Introducing $\mathbf{u} = u\hat{\mathbf{x}}$, and $\hat{\mathbf{k}} \cdot \hat{\mathbf{x}} = \cos\theta_k$, yields the

explicit form

$$\hat{\mathbf{e}}(\hat{\mathbf{k}}) = 2\hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}) - \hat{\mathbf{x}} = (2 \cos^2 \theta_k - 1)\hat{\mathbf{x}} + 2 \sin \theta_k \cos \theta_k \hat{\mathbf{y}} = -\cos(2\theta_k)\hat{\mathbf{x}} + \sin(2\theta_k)\hat{\mathbf{y}} \quad , \quad (10)$$

which clearly sweeps out one complete unit circle, clockwise, as θ_k goes from $-\pi/2$ to $+\pi/2$.

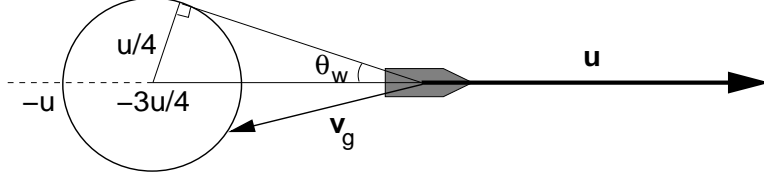


Figure 2: The locus of group velocity vectors, $\mathbf{v}_g(\mathbf{k})$, from a ship moving at velocity \mathbf{u} .

Based on the discussion above we can read eq. (9) geometrically in the manner illustrated in fig. 2. The group velocities of stationary waves lie on a circle of radius $u/4$ centered at $-3\mathbf{u}/4$. It is evident from this geometric construction that energy moves away from the ship in a restricted range of *sternward* directions, even as the waves vectors themselves span all possible *forward* directions. Constructing a triangle using the radius and tangent (see fig. 2) we find that the circle subtends a half-angle

$$\theta_w = \sin^{-1}(1/3) = 19.47^\circ \quad , \quad (11)$$

behind the ship. The wake is therefore a wedge making a complete angle $2\theta_w = 39^\circ$ behind the ship independent of speed.

Which wave vector moves at the limiting group velocity making the angle $\theta_w = 19.47^\circ$? To answer this question we consider a similar triangle using a unit circle, as shown in fig. 3. From this we see that

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{e}}(\hat{\mathbf{k}}_{\text{cr}}) = \cos \alpha = \sin \theta_w = \frac{1}{3} \quad . \quad (12)$$

Using eq. (10) in this shows that $2\theta_k = \pi - \alpha = \pi/2 - \theta_w$ and thus that $\theta_k = 45^\circ - 19.47^\circ/2 = 35.26^\circ$. The relation $2\theta_k = \pi/2 - \theta_w$ also means that the wave fronts, perpendicular to $\hat{\mathbf{k}}$, make the same angle $\theta_k = 35.26^\circ$ with the wake itself, as shown in fig. 3. Finally we can use these expressions to provide

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{k}} = \cos \theta_k = \sqrt{\frac{1}{2}[\cos(2\theta_k) + 1]} = \sqrt{\frac{1}{2}[\sin(\theta_w) + 1]} = \sqrt{\frac{2}{3}} \quad . \quad (13)$$

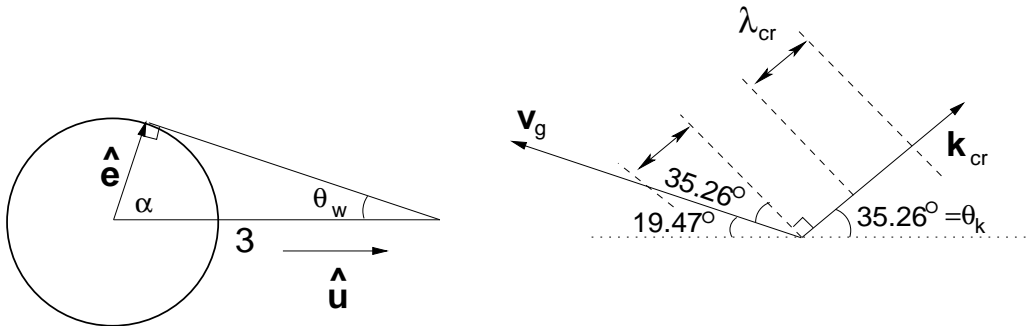


Figure 3: The geometric construction used to find the limiting wave vector, \mathbf{k}_{cr} .

Using eq. (13) in the expression (5) gives the limiting wave vector $k_{\text{cr}} = 3g/2u^2$. The crests of this wave are separated by $\lambda_{\text{cr}} = 4\pi u^2/3g$. The crests along the edge of the wake are therefore separated by

$$\ell_{\text{cr}} = \frac{\lambda_{\text{cr}}}{\sin(\theta_k)} = \frac{4\pi u^2}{\sqrt{3}g} = 7.25 \frac{u^2}{g} \quad . \quad (14)$$

The crest spacing depends on the ship's speed even while the angle of the wake does not.

Kelvin's method

In a general super-position of surface waves each wave vector appears with a complex amplitude, $A(\mathbf{k})$, depending on, among other things, the shape of the ship's hull

$$z(\mathbf{x}, t) = \int A(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega(\mathbf{k})t} d\mathbf{k} , \quad (15)$$

where $\mathbf{x} = (x, y)$ is a position on the surface. This can be restricted this to stationary waves, $\omega = 0$, using expression (5) to write $\mathbf{k} = \hat{\mathbf{k}}g/(\mathbf{u} \cdot \hat{\mathbf{k}})^2$.

$$z(r, \phi) = \int A(\theta_k) \exp \left[\frac{ig \hat{\mathbf{k}} \cdot \mathbf{x}}{(\mathbf{u} \cdot \hat{\mathbf{k}})^2} \right] d\theta_k = \int A(\theta_k) \exp \left[i \frac{rg}{u^2} \psi(\phi, \theta_k) \right] d\theta_k , \quad (16)$$

where

$$\psi(\phi, \theta_k) = \frac{\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}}{(\hat{\mathbf{u}} \cdot \hat{\mathbf{k}})^2} = \frac{\cos(\phi - \theta_k)}{\cos^2 \theta_k} , \quad (17)$$

after taking $\hat{\mathbf{u}} = \hat{\mathbf{x}}$. The integral runs from $\theta_k = -\pi/2$ to $\pi/2$ using wave vectors satisfying $\omega_+ = 0$ — i.e. the blue curve in fig. 1. It then completes the circle from $\theta_k = \pi/2$ to $3\pi/2$ using stationary waves from the other branch, $\omega_- = 0$. The magnitudes of both branches satisfy eq. (5), so the distinction can be dropped from the integral in eq. (16).

In the limit $rg/u^2 \gg 1$ the exponential factor varies extremely rapidly and will average away contributions to $z(r, \phi)$ except those by wave vectors with “stationary phase”

$$\frac{\partial \psi}{\partial \theta_k} = \frac{\sin(\phi - \theta_k)}{\cos^2 \theta_k} + 2 \frac{\cos(\phi - \theta_k) \sin \theta_k}{\cos^3 \theta_k} = 0 . \quad (18)$$

The requirement for stationary phase thereby associates with any spatial angle ϕ several wave-vector directions θ_k . The two are related through

$$\tan(\theta_k - \phi) = 2 \tan \theta_k , \quad (19)$$

from a manipulation of eq. (18).

Equation (19) is solved by setting $\phi = \theta_g$ the angle of the group velocity, $\mathbf{v}_g(\hat{\mathbf{k}})$ in eq. (6). To see this take the dot product and cross product of this equation with $\hat{\mathbf{k}}$. The result is

$$\mathbf{v}_g \cdot \hat{\mathbf{k}} = v_g \cos(\theta_k - \theta_g) = -\frac{1}{2} \mathbf{u} \cdot \hat{\mathbf{k}} = -\frac{1}{2} u \cos(\theta_k) \quad (20)$$

$$\mathbf{v}_g \times \hat{\mathbf{k}} = v_g \sin(\theta_k - \theta_g) \hat{\mathbf{z}} = -\mathbf{u} \times \hat{\mathbf{k}} = -u \sin(\theta_k) \hat{\mathbf{z}} . \quad (21)$$

Taking a simple ratio of these expressions returns eq. (19) immediately. There is, moreover, a 180° ambiguity in the choice of ϕ satisfying eq. (19), stemming from the periodicity of \tan . Since a general superposition of stationary waves is acausal it might be arranged to produce a disturbance *in front* of a ship as easily as *behind* it. This ambiguity is absent from the group velocity, which is confined to directions behind the ship.

Equation (19) can be solved to yield a closed form expression for the spatial direction for which a particular wave vector direction is stationary

$$\phi(\theta_k) = \theta_k - \tan^{-1} [2 \tan \theta_k] . \quad (22)$$

From the foregoing discussion we see that appropriate branch of \tan^{-1} is the one centered on -180° in order to produce the graph shown in fig. 4. The plot shows that ϕ ranges within the wedge of angles $180^\circ \pm \theta_w$. It achieves these extrema at $\theta_k = \pm 35.26^\circ$ and $\pm 144.74^\circ$ (squares). Outside this wedge of angles there is no point in the integral, eq. (16), with stationary phase. This means that outside the wake the different contributions mostly average away at large distances, $r \gg u^2/g$.

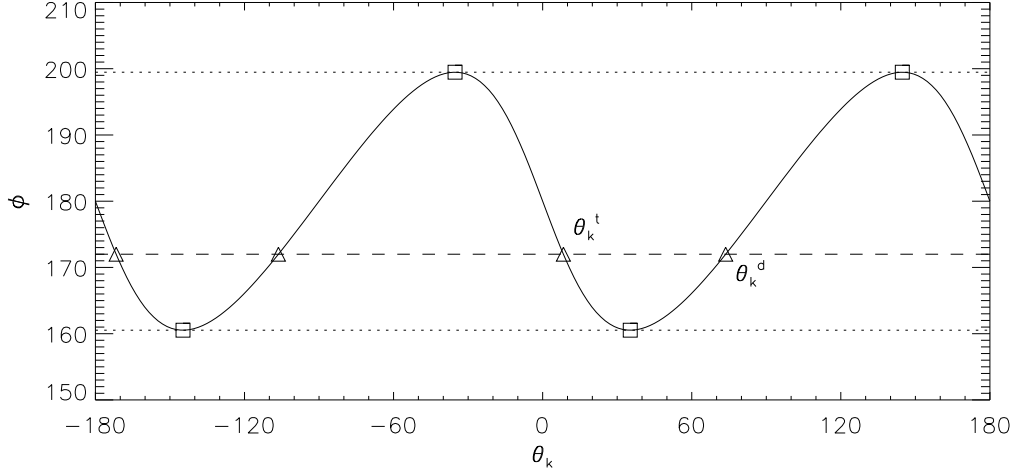


Figure 4: Plot of $\phi(\theta_k)$ from eq. (4).

For a given value of ϕ between extremes (dashed line in fig. 4) the functional relationship, (22), can be inverted to yield four different wave vector angles (triangles): $\theta_k^t(\phi)$ and $\theta_k^d(\phi)$ and their $-\mathbf{k}$ counterparts $\theta_k^t(\phi) - 180^\circ$ and $\theta_k^d(\phi) - 180^\circ$. The integral in eq. (16) can be approximated as a sum over contributions from these four wave vectors where the integrand does not average away. We may also note that since $z(x, y)$ is real $A(\theta_k - 180^\circ) = A^*(\theta_k)$. Furthermore, from eq. (17) we find that $\psi(\phi, \theta_k - 180^\circ) = -\psi(\phi, \theta_k)$. This allows us to write the wake as a sum over two wave vectors

$$z(r, \phi) \simeq 2\text{Re} \{ A(\theta_k^t) \exp[i(rg/u^2)\psi(\phi, \theta_k^t)]/C_t \} + 2\text{Re} \{ A(\theta_k^d) \exp[i(rg/u^2)\psi(\phi, \theta_k^d)]/C_d \} ,$$

where C_t and C_d come from the stationary phase integral.

The expression is simplified, and clarified, if we assume that $A/C = \tilde{A}$ is purely real

$$z(r, \phi) \sim 2\tilde{A}_t \cos [(rg/u^2)\psi(\phi, \theta_k^t)] + 2\tilde{A}_d \cos [(rg/u^2)\psi(\phi, \theta_k^d)] . \quad (23)$$

In the far field, $r \gg u^2/g$, the cosine factors will vary rapidly, while the amplitudes \tilde{A} vary more slowly. Each contribution consists of oscillations which crest where the argument of its cosine is an integer multiple of 2π . The crests from each component thus form a series of curves

$$r(\phi) = 2\pi n \frac{u^2}{g \psi[\phi, \theta_k(\phi)]} = 2\pi n \frac{u^2}{g} \frac{\cos^2[\theta_k(\phi)]}{\cos[\theta_k(\phi) - \phi]} , \quad (24)$$

after using eq. (17). The two inverse branches, θ_k^t and θ_k^d , yield two different crest curves called transverse (t) and divergent (d), plotted as solid curves in figure 5. Recall that both inverses $\theta_k(\phi)$ are undefined for arguments outside the wake, so the crest curves do not extend there.

The inverses for $\phi = 180^\circ$ are $\theta_k^t(180^\circ) = 0$ and $\theta_k^d(180^\circ) = 90^\circ$, yielding crests at $r_t = -2\pi n u^2/g$ and $r_d = 0$, respectively. In other words the divergent crests do not cross behind the ship while the transverse crests do. Both crest curves meet at the edge of the wake, $\phi = 180^\circ - \theta_w$, where $\theta_k^t = \theta_k^d = \theta_k = \cos^{-1}(\sqrt{2/3})$ from eq. (13). Using the fact that $2\theta_k = 90^\circ - \theta_w$ we find

$$\theta_k - \phi = \theta_k + \theta_w - 180^\circ = -90^\circ - \theta_k .$$

We can use this to write the crest locations in the form

$$r(180^\circ - \theta_w) = -2\pi n \frac{u^2}{g} \frac{\cos^2(\theta_k)}{\sin(\theta_k)} = -2\pi n \frac{u^2}{g} \frac{(2/3)}{1/\sqrt{3}} = -\frac{4\pi u^2}{\sqrt{3}g} n = -7.25 \frac{u^2}{g} n , \quad (25)$$

the same spacing derived in eq. (14).

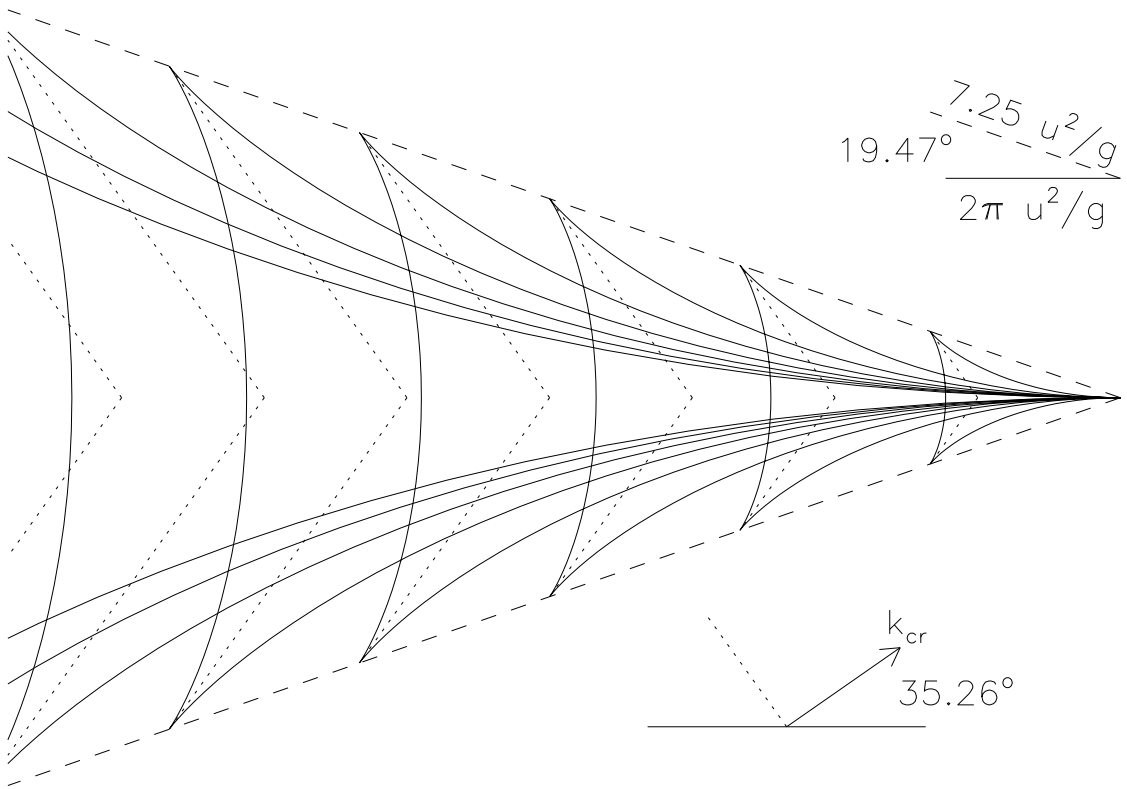


Figure 5: The curves of transverse and divergent crests.