

A SUMMARY OF USEFUL 2D MAGNETIC FIELDS

Complex potentials: $A_z(x, y) = \text{Re}\{\hat{F}(x + iy)\}$

$$B_y + iB_x = -\hat{F}'(w)$$

$$\text{wire: } \hat{F}(w) = -\frac{4\pi I \ln w}{c \cdot 2\pi}$$

Green-Syrovatskii CS of length $\Delta = 2\delta$

$$\hat{F}(w) = \frac{1}{2}B'w\sqrt{w^2 - \delta^2} - \frac{1}{2}B'\delta^2 \ln(w + \sqrt{w^2 - \delta^2})$$

$$B_y + iB_x = -B'\sqrt{w^2 - \delta^2}$$

$$\frac{4\pi I}{c} = \pi B'\delta^2 = \frac{\pi}{4}B'\Delta^2$$

Axisymmetric fields: $\mathbf{B} = \nabla f \times \nabla \phi$

$$\Phi(\varpi < \rho) = 2\pi f(\rho, z)$$

$$\nabla \times \mathbf{B} = \varpi^2 \nabla \cdot \left(\frac{\nabla f}{\varpi^2} \right) \nabla \phi \equiv \Delta^* f \nabla \phi$$

$$= \left[\frac{\partial^2 f}{\partial r^2} + \frac{1 - \mu^2}{r^2} \frac{\partial^2 f}{\partial \mu^2} \right] \nabla \phi$$

$$= \left[\varpi \frac{\partial}{\partial \varpi} \left(\frac{1}{\varpi} \frac{\partial f}{\partial \varpi} \right) + \frac{\partial^2 f}{\partial z^2} \right] \nabla \phi$$

2^p-pole potential fields: $f_p(r, \mu)$

$$f_0 = 1 - \mu = 1 - \frac{z}{\sqrt{\varpi^2 + z^2}}$$

$$f_1 = \frac{1 - \mu^2}{r} = \frac{\varpi^2}{(\varpi^2 + z^2)^{3/2}}$$

$$f_2 = \frac{\mu(1 - \mu^2)}{3r^2} = \frac{z\varpi^2}{3(\varpi^2 + z^2)^{5/2}}$$

Current ring @ $(\varpi, z) = (\rho, 0)$

$$f(\varpi, z) = \frac{4\pi I}{c} \frac{\sqrt{\rho\varpi}}{k} [(2 - k^2)K(k) - 2E(k)]$$

$$k^2 = \frac{4\rho\varpi}{(\varpi + \rho)^2 + z^2}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \cos^2 \theta}}$$

$$\simeq \frac{1}{2} \ln \left(\frac{16}{1 - k^2} \right), \quad k \rightarrow 1$$

Force-free equilibria:

Lundquist field (const.- α)

$$\mathbf{B} = B_0 [J_0(\alpha\varpi)\hat{\mathbf{z}} + J_1(\alpha\varpi)\hat{\phi}], \quad \nabla \times \mathbf{B} = \alpha\mathbf{B}$$

$$\Phi(\varpi < \rho) = \frac{2\pi\rho B_0}{\alpha^2} J_1(\alpha\rho)$$

$$\Phi(\varpi < j_{0,1}/\alpha) = 0.431 \pi (j_{0,1}/\alpha)^2 B_0$$

Gold-Hoyle field (const. pitch $q = B_\phi/\varpi B_z$)

$$\mathbf{B} = \frac{B_0(\hat{\mathbf{z}} + q\varpi\hat{\phi})}{1 + q^2\varpi^2}, \quad \nabla \times \mathbf{B} = \frac{2q\mathbf{B}}{1 + q^2\varpi^2}$$

$$\Phi(\varpi < \rho) = \frac{\pi B_0}{q^2} \ln(1 + q^2\rho^2)$$

$$\frac{4\pi I}{c}(\varpi < \rho) = \frac{2\pi B_0}{q} \frac{q^2\rho^2}{1 + q^2\rho^2}$$

Axisymmetric constant- α field:

$$\mathbf{B} = \nabla f \times \nabla \phi + \alpha f \nabla \phi, \quad B_\phi = \alpha f / \varpi$$

$$\Delta^* f = -\alpha^2 f$$

2^p-pole solutions:

$$f_0 = -(1 - \mu)\alpha r n_0(\alpha r) = (1 - \mu) \cos(\alpha r)$$

$$f_1 = -(1 - \mu^2)\alpha^2 r n_1(\alpha r)$$

$$= (1 - \mu^2) \left[\frac{\cos(\alpha r)}{r} + \alpha \sin(\alpha r) \right]$$

$$f_2 = -\mu(1 - \mu^2)\alpha^3 r n_2(\alpha r)$$

Point-source: (Chiu & Hilton, 1977)

$$f = \cos(\alpha\mu r) - \mu \cos(\alpha r)$$

Spheromak: $\nabla \times \mathbf{B} = \alpha = 0, \quad r > a$

$$f = \begin{cases} f_0 \frac{\xi r}{a} j_1\left(\frac{\xi r}{a}\right) \sin^2 \theta, & r \leq a \\ f_0 C \xi \left(\frac{a}{r} - \frac{r^2}{a^2}\right) \sin^2 \theta & r > a \end{cases}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_1(\xi) = 0 = \tan \xi - \xi$$

$$\xi = 4.49341, \quad C = -\frac{1}{3} \sin \xi = 1.46204$$

$$\alpha = \xi/a, \quad \text{axis: } r_0 = 0.61067a$$