Inferring a Photospheric Velocity Field from a Sequence of Vector Magnetograms: The Minimum Energy Fit†

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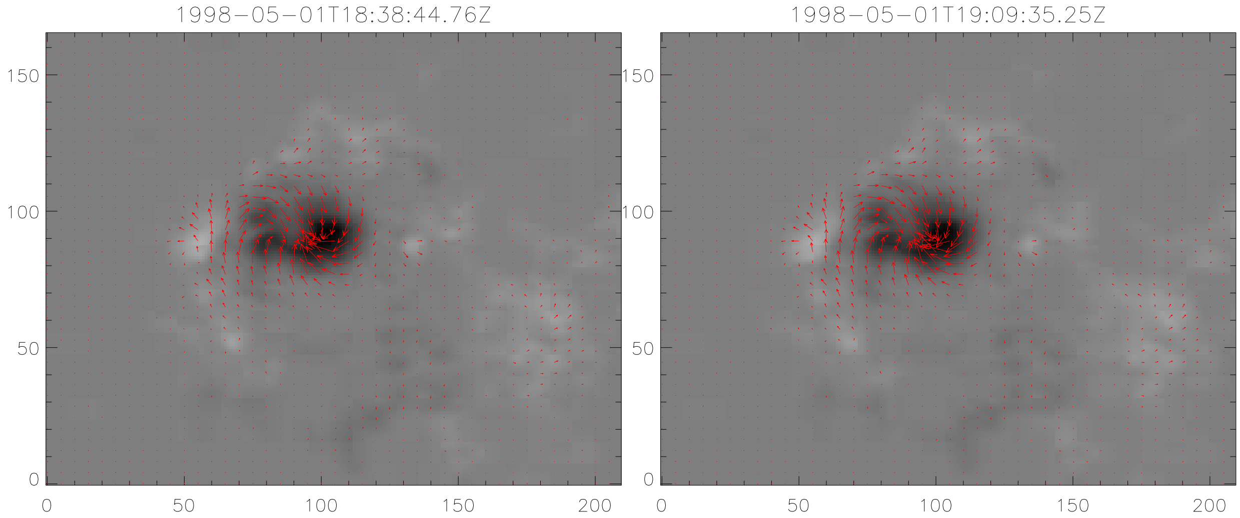
"WHAT ARE WE UP TO?"

- From a sequence (pair) of vector magnetograms...



IVM m-grams of AR 8210: 1998 May 1

- Deduce a* photospheric 3-component velocity field consistent w/ their evolution

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† This work was funded by the Department of Defense through its MURI grant, "Understanding Magnetic Eruptions on the Sun and their Interplanetary Consequences", administered by the AFOSR.

∗ We show that the m-gram evolution is consistent w/ ∞ly many velocity fields. Must choose one.

"HOW CAN IT BE DONE?": THE METHOD

- Ideal Induction equation at p-sphere ($z = 0$):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \qquad \text{(Induc. Eq.)}$$

- Vertical component¶ depends on $\mathbf{v}$ and $\mathbf{B}$ within $z = 0$ plane

$$\frac{\partial B_z}{\partial t} = \nabla_h \cdot (v_z \mathbf{B}_h - B_z \mathbf{v}_h) \qquad (\dagger)$$

- Unknown velocity from unknown potentials* $\phi(x, y)$ and $\psi(x, y)$

$$v_z \mathbf{B}_h - B_z \mathbf{v}_h = \nabla_h \phi + \nabla_h \psi \times \hat{\mathbf{z}}$$

- Equation (†) fixes $\phi(x, y)$

$$\nabla_h^2 \phi = \frac{\partial B_z}{\partial t} \qquad (\phi - \text{eq})$$

$*$ ϕ & ψ: Related to horizontal E-field:

$$\mathbf{E}_h = -\hat{\mathbf{z}} \times (v_z \mathbf{B}_h - B_z \mathbf{v}_h) = \nabla \phi \times \hat{\mathbf{z}} - \nabla_h \psi \ ,$$

¶ Horizontal components of (Induc. Eq.):

$$\frac{\partial \mathbf{B}_h}{\partial t} = \cdots + \underbrace{\mathbf{B}_h \frac{\partial v_z}{\partial z} + B_z \frac{\partial \mathbf{v}_h}{\partial z}}_{\text{more unknowns}} - v_z \underbrace{\frac{\partial \mathbf{B}_h}{\partial z}}_{\dagger\dagger}$$

†† Vertical gradient *not yet* observable — dual-level phot/chrom or high-photosphere m-grams may yield this (Leka & Metcalf 2003).

THE METHOD (cont'd)

$$\frac{\partial B_z}{\partial t} \;=\; \nabla_h \cdot (v_z \mathbf{B}_h - B_z \mathbf{v}_h) \;=\; \nabla_h^2 \phi \qquad (\dagger)$$

- ∘ $\phi(x,y)$ found $\Longrightarrow$ $\hat{\mathbf{z}}\cdot$(Induc. Eq.) satisfied
- ∘ $\psi(x,y)$ and v_z are completely unconstrained by ($\dagger$)
- ∘ Choose from ∞ possibilities by minimizing functional $W\{\psi, v_z\}$
- ∘ Design functional to yield…

1. Smallest r.m.s. velocity

$$W\{\psi, v_z\} \;\equiv\; \tfrac{1}{2}\int \left[\, |\mathbf{v}_h|^2 + |v_z|^2 \right] dx\, dy$$

2. Closest to some imperfectly known* velocity $\mathbf{u}(x,y)$

$$W\{\psi, v_z\} \;\equiv\; \tfrac{1}{2}\int \left[\, |\mathbf{v}_h - \mathbf{u}_h|^2 + |v_z - u_z|^2 \right] dx\, dy$$

$\rightarrow$ Strategy 1: a special case of strategy 2: $\mathbf{u} = 0$.

$*$ Possible sources of independent velocity information:

- $u_{\rm los}(x,y)$ from Doppler measurements
- $\mathbf{u}_h(x,y)$ from Local Correlation Tracking (November & Simon 1988).
- $\mathbf{u}(x,y)$ from last step in sequential inversions — $\mathbf{v}(x,y,t)$ would be "least changing" solution

THE VARIATION

- Substitute (ϕ, B_z, $\mathbf{B}_h$ known — ψ and v_z unknown)

$$\mathbf{v}_h \;=\; \frac{1}{B_z}\left(v_z\mathbf{B}_h - \nabla_h\phi - \nabla_h\psi \times \hat{\mathbf{z}}\right)$$

- …into

$$W\{\psi, v_z\} \;\equiv\; \tfrac{1}{2}\int \Big[\,|\mathbf{v}_h - \mathbf{u}_h|^2 + |v_z - u_z|^2\Big]\; dx\, dy$$

- Vary $\psi(x,y)$ — hold $v_z(x,y)$ fixed

$$\nabla_h\cdot\left(\frac{\nabla_h\psi}{B_z^2}\right) \;=\; \nabla_h\cdot\left[\frac{\hat{\mathbf{z}} \times (v_z\mathbf{B}_h - \nabla_h\phi - B_z\mathbf{u}_h)}{B_z^2}\right] \qquad \text{(EL1)}$$

- Vary $v_z(x,y)$ — hold $\psi(x,y)$ fixed

$$v_z \;=\; \frac{B_z^2\, u_z \;+\; \mathbf{B}_h\cdot(\nabla_h\phi + \nabla_h\psi \times \hat{\mathbf{z}} + B_z\mathbf{u}_h)}{|\mathbf{B}|^2} \qquad \text{(EL2)}$$

Solution satisfying (EL1) & (EL2) together is
The Minimum Energy Fit (MEF)

FACTS ABOUT THE MEF SOLUTION:

- Substitute (EL2) $\longrightarrow$ (EL1): Elliptic Eq. for $\psi(x,y)$

$$\nabla_h \cdot \left(\frac{\Pi \cdot \nabla_h \psi}{B_z^2} \right) = -\nabla_h \cdot \left[\frac{\Pi \cdot (\hat{\mathbf{z}} \times \nabla_h \phi)}{B_z^2} \right]$$

$$\Pi \;\equiv\; I \;-\; \frac{(\hat{\mathbf{z}} \times \mathbf{B}_h)(\hat{\mathbf{z}} \times \mathbf{B}_h)}{|\mathbf{B}_h|^2 + B_z^2}$$

Π projects $\perp$ to $\mathbf{B}_h$ @ PIL ($B_z = 0$)

- Combining

$$\mathbf{v}_h \;=\; \frac{1}{B_z}\,(v_z \mathbf{B}_h - \nabla_h \phi - \nabla_h \psi \times \hat{\mathbf{z}})$$

$$v_z \;=\; \frac{B_z^2\, u_z \;+\; \mathbf{B}_h \cdot (\nabla_h \phi + \nabla_h \psi \times \hat{\mathbf{z}} + B_z \mathbf{u}_h)}{|\mathbf{B}|^2}$$

1. Without indep. flow info.— ($\mathbf{u} = 0$)

   $$\mathbf{v} \cdot \mathbf{B} \;=\; v_z B_z \;+\; \mathbf{v}_h \cdot \mathbf{B}_h \;=\; 0 \quad \text{i.e. } \mathbf{v}_{\|} = 0$$

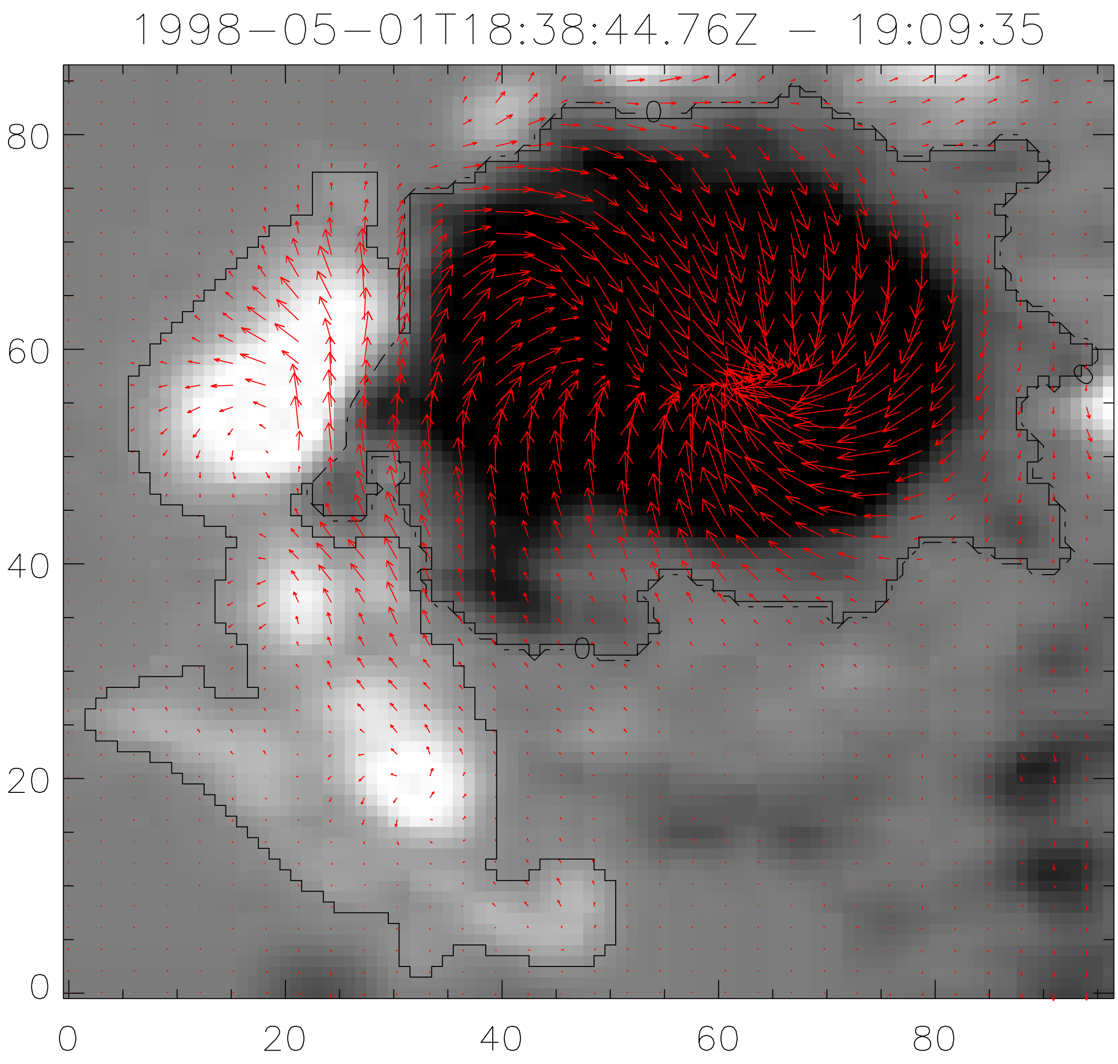
   (Induc. Eq.) involves only $\mathbf{v} \times \mathbf{B} \Longrightarrow \mathbf{v}_{\|} = 0$ naturally

2. With indep. flow info. $\mathbf{u}(x,y)$

   $$(\mathbf{v} - \mathbf{u}) \cdot \mathbf{B} \;=\; 0$$

   in directions along field $\mathbf{v}_{\|} = \mathbf{u}_{\|}$

"HOW DOES IT WORK?"
THE IMPLEMENTATION:



- Combine m-grams $\mathbf{B}^{(1)}(x,y)$ and $\mathbf{B}^{(2)}(x,y)$:

$$\frac{\partial B_z}{\partial t} = \frac{B_z^{(2)} - B_z^{(1)}}{t_2 - t_1} \quad , \quad \bar{\mathbf{B}} \; = \; \tfrac{1}{2}B_z^{(2)} + \tfrac{1}{2}B_z^{(1)}$$

- Create contiguous mask ($\mathbf{v} = 0$ outside)

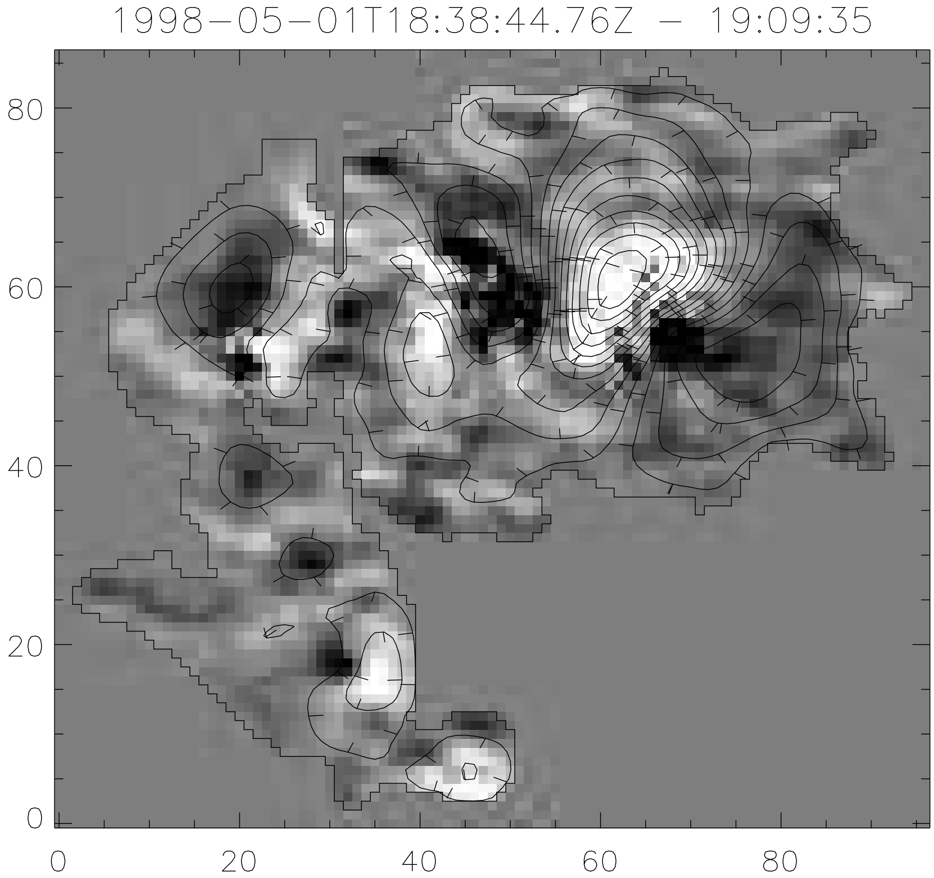
$$M(x,y) \; = \; \begin{cases} 1 & , \quad |\bar{B}_z| > 3\sigma_{B_z} \\ 0 & , \quad \text{everywhere else} \end{cases}$$

THE IMPLEMENTATION: (cont'd)

∘ Solve (ϕ-eq) inside M — relaxation method

∘ BC: $\phi = 0$ on boundary ∂M

$$\nabla_h^2 \phi \;=\; \frac{\Delta B_z}{\Delta t}$$



Grey-scale:$\Delta B_z/\Delta t$; contours: $\phi(x, y)$

$$v_z \mathbf{B}_h - B_z \mathbf{v}_h \;=\; \nabla_h \phi \;+\; \nabla_h \psi \times \hat{\mathbf{z}}$$

$$\left[\quad \mathbf{E}_h \;=\; -\,\hat{\mathbf{z}} \times (v_z \mathbf{B}_h - B_z \mathbf{v}_h) \;=\; \nabla \phi \times \hat{\mathbf{z}} \;-\; \nabla_h \psi \quad\right]$$

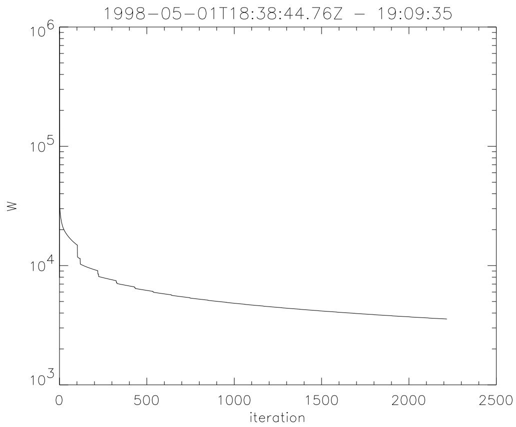
THE IMPLEMENTATION: (cont'd)

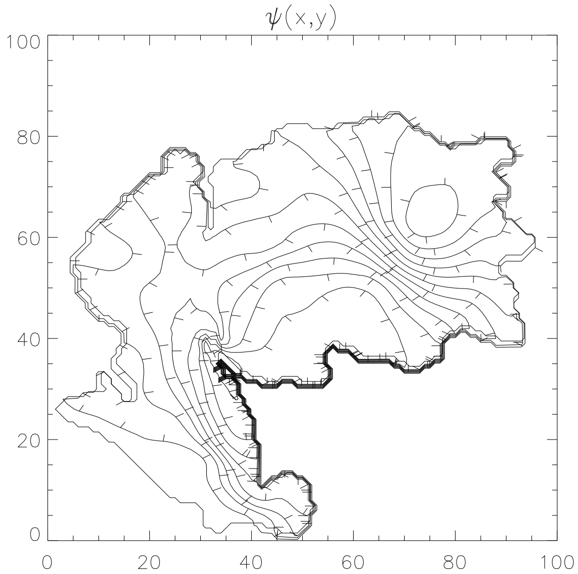
∘ Solve (EL1) & (EL2) by alternating relaxations —

1. Solve* (EL1) for $\psi^{(0)}(x,y)$ w/ $v_z = 0$
   — Jacobi relaxation
2. Solve† (EL2) for $v_z^{(1)}(x,y)$ using $\psi = \psi^{(0)}$
3. Solve* (EL1) for $\psi^{(1)}(x,y)$ w/ $v_z = v_z^{(1)}$
2. Solve† (EL2) for $v_z^{(2)}(x,y)$ using $\psi = \psi^{(1)}$

$$\vdots$$

n. Repeat until changes are "small enough"



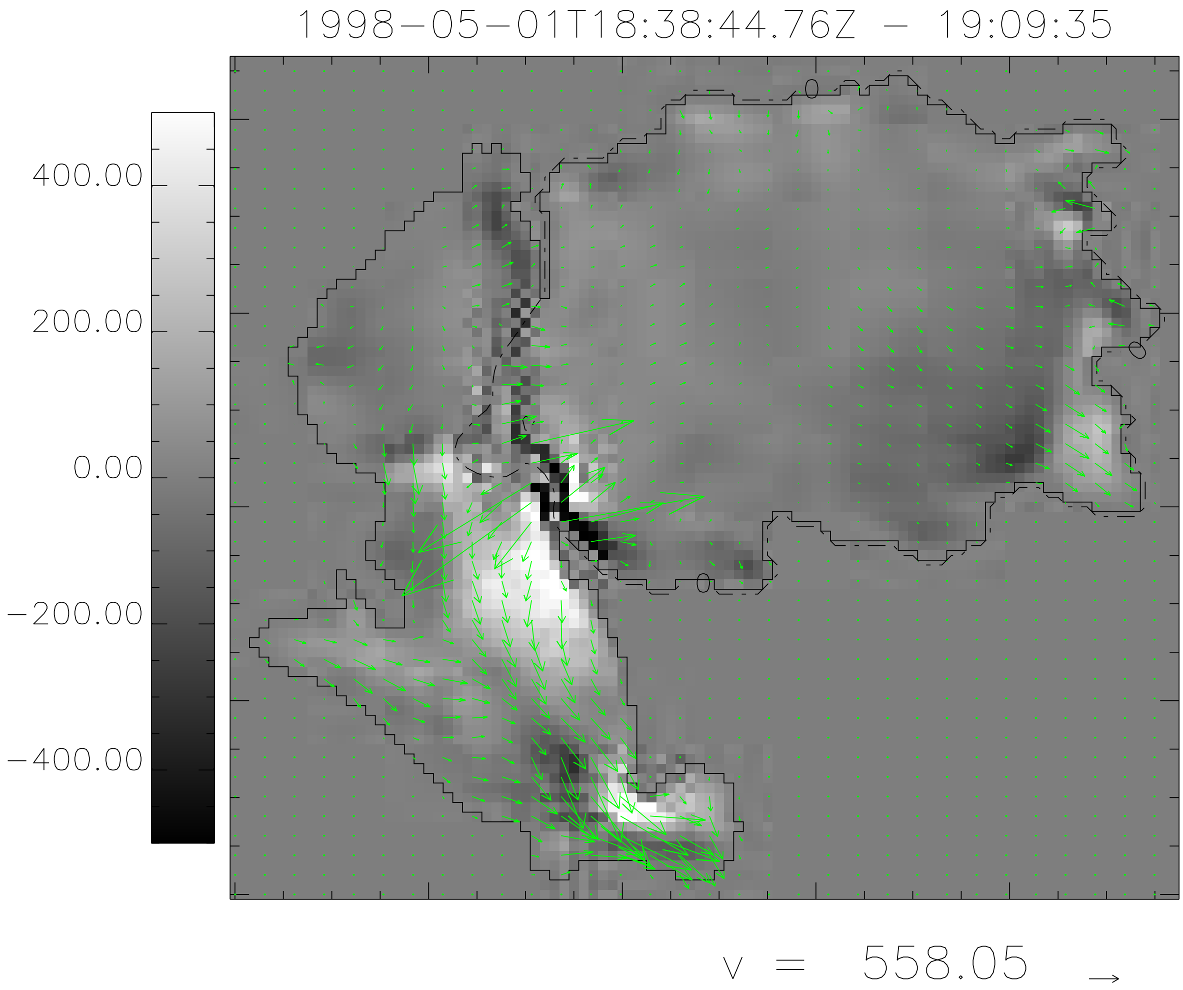


∗ Solve (EL1) in M w/ BC: $\mathbf{v}_\perp \cdot \hat{\mathbf{n}} = 0$ on ∂M

† Minimizing $W(\psi_{i,j}, w_{i,j})$ yields coupled equations
— solve by relaxation

THE IMPLEMENTATION: (cont'd)

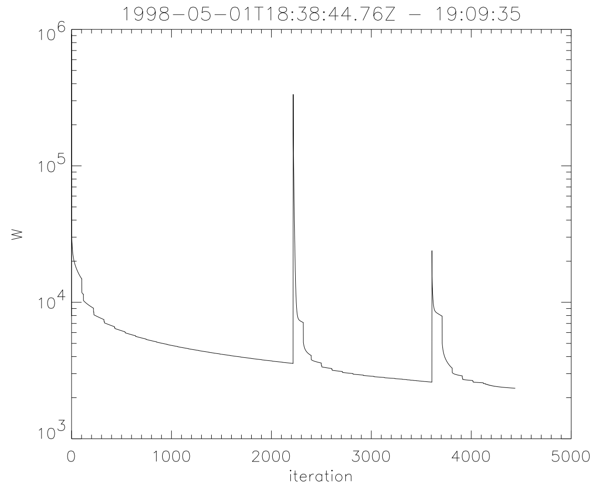
- Result of relaxation: Contains artifacts along PIL:

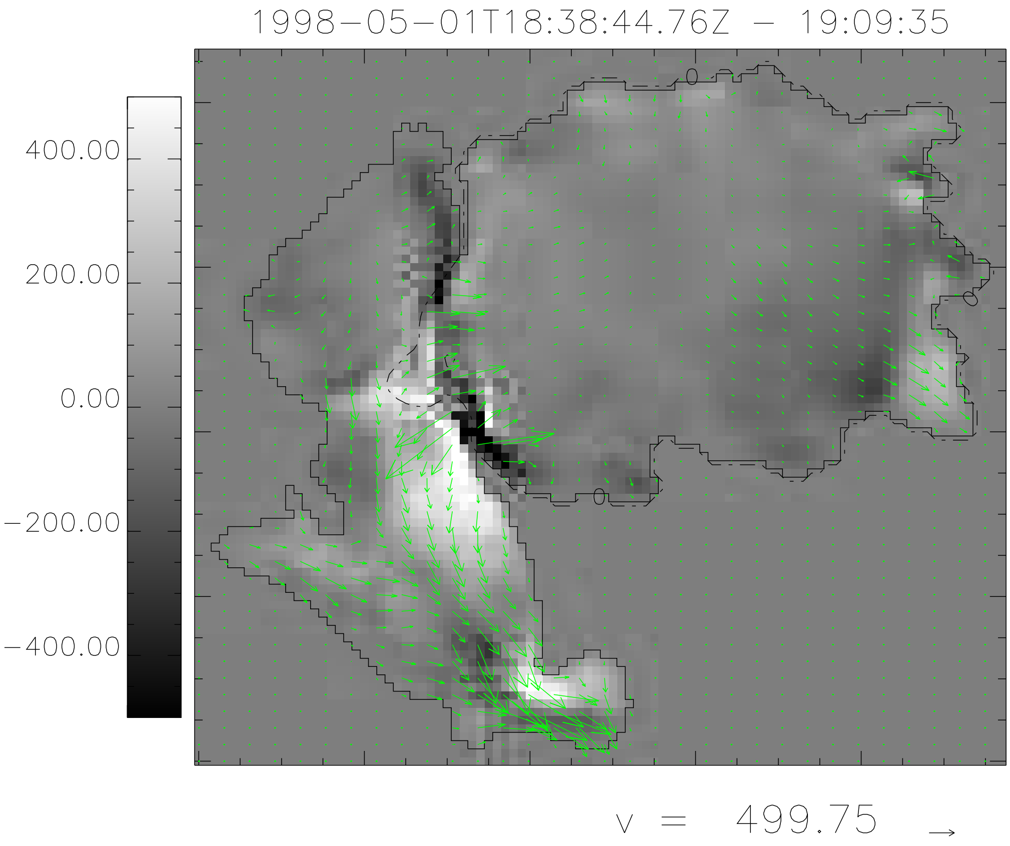


- Probable cause(s)
  - Poor convergence of v_z relaxation @ PIL
  - Poor coupling across PIL due to nature of Π operator
- Fix: smooth v_z — repeat relaxation…

THE IMPLEMENTATION: (cont'd)

- smooth & relax as necessary

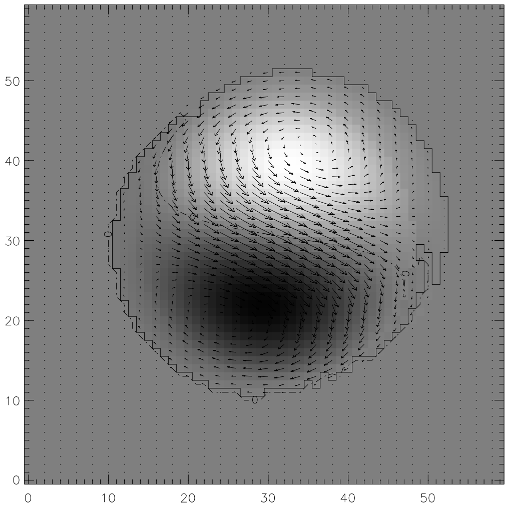


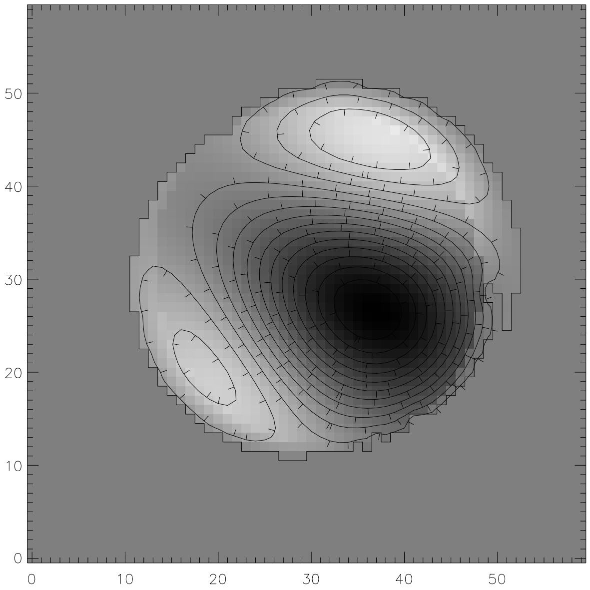


"DOES IT REALLY WORK?" A TEST CASE

- Spheromak: radius 20, center $\mathbf{x}^{(c)} = (0, 0, -3)$

Slice at $z = 0$: $\mathbf{B}^{(1)}(x, y)$
Move $\mathbf{x}^{(c)}$ by $\mathbf{v} = (3.3, 2.3, 4.0)$
(uniform translation)





$\mathbf{v}_\perp$: ($\perp$ to $\mathbf{B}$)
$\Longrightarrow$ "Generating flow"
MEF $\Longrightarrow$ "inferred flow"

- Compare

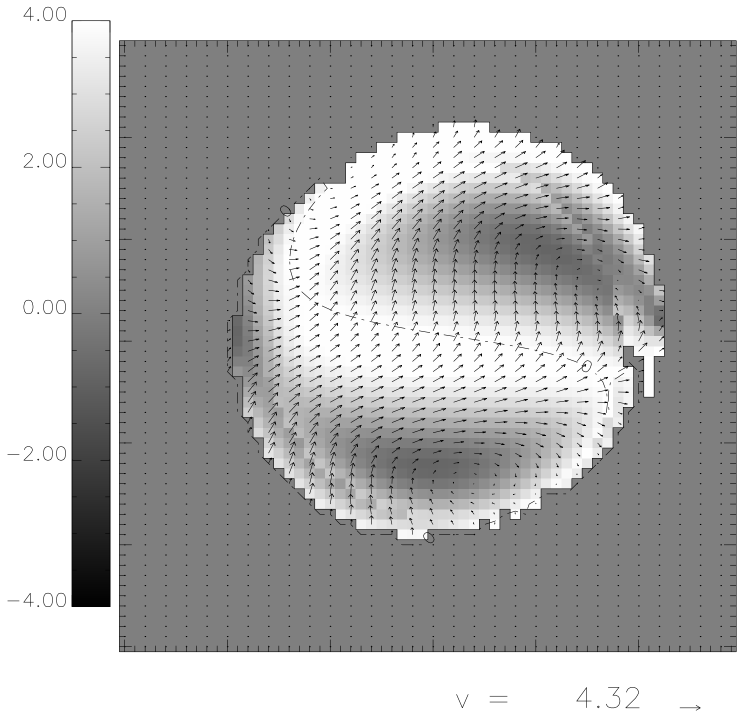
$$\text{Mass flux} \;:\; F \;=\; \int_M v_z \, dx \, dy$$

$$\text{Poynting Flux} \;:\; P \;=\; \frac{1}{4\pi} \int_M |\mathbf{B}|^2 \mathbf{v}_\perp \cdot \hat{\mathbf{z}} \, dx \, dy$$

A TEST CASE (cont'd)

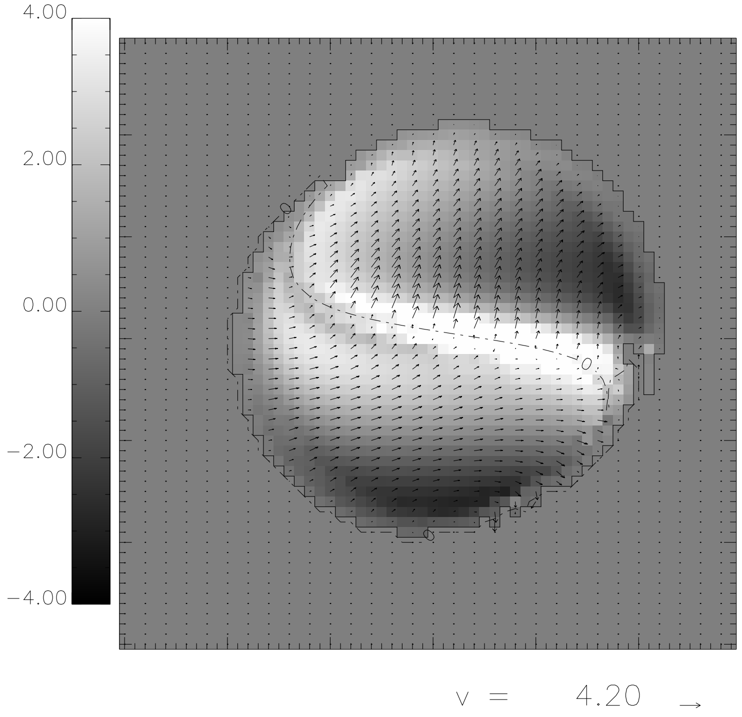
Generating flow:

$W_{\perp} = 2.7 \times 10^4$
$F = 3.1 \times 10^3$
$P = 9.4 \times 10^3$



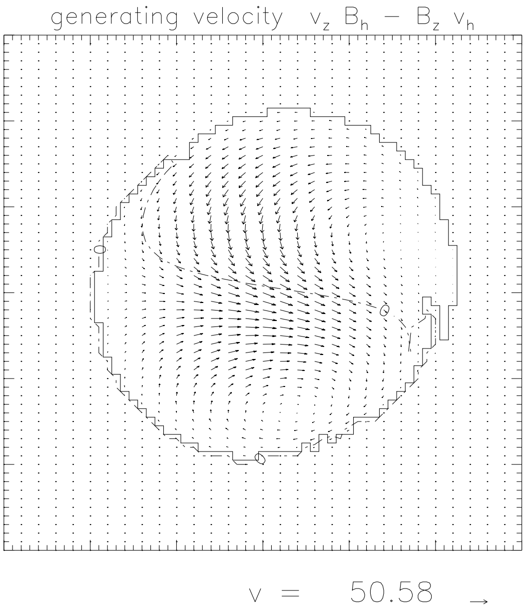
Result of MEF:

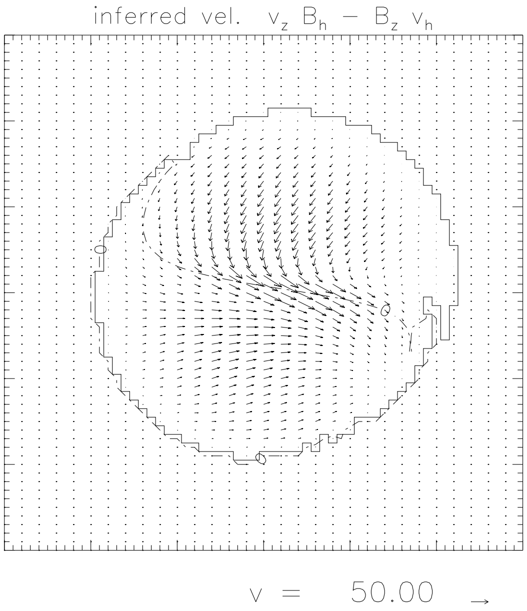
$W = 1.86 \times 10^4$
$F = 1.1 \times 10^3$
$P = 6.2 \times 10^3$



A TEST CASE (cont'd)

∘ Compare $\mathbf{f} = v_z\mathbf{B}_h - B_z\mathbf{v}_h = \nabla_h\phi + \nabla_h\psi\times\hat{\mathbf{z}} = \mathbf{E}_h\times\hat{\mathbf{z}}$

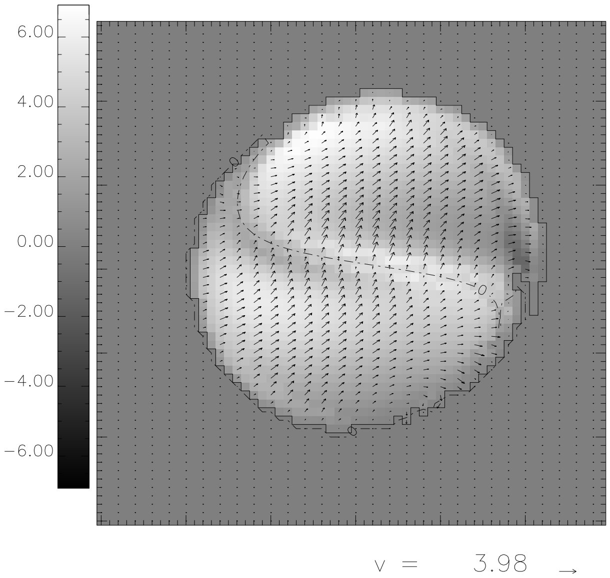




∘ Solve w/ additional information: $\mathbf{u} = (0, 0, 4)$

Inferred Flow:
$F = 4.49 \times 10^3$
$W = 1.57 \times 10^4$

Generating Flow
$F_0 = 5.14 \times 10^3$
$W_0 = 2.06 \times 10^4$



"IS THE MEF LIKE LOCAL CORRELATION TRACKING?"

- Local Correlation tracking (LCT, November & Simon 1988) uses the evolution of an image $I(x, y, t)$ to find horizontal velocity $\mathbf{u}_h(x, t)$
- Forms cross-correlation between successive images using *apodizing window* of width w, centered at position $\mathbf{x}_0$.
- Cross-corr'n peak @ $\Delta\mathbf{x} \implies \mathbf{u}_h(x_0, y_0) = \Delta\mathbf{x}/\Delta t$
  Assuming:

  1. $I(x, y, t)$ solves horizontal advection Eq.

  $$\frac{\partial I}{\partial t} + \mathbf{u}_h \cdot \nabla I \; = \; 0$$

  2. $\mathbf{u}_h(x, y)$ is structure only on scales $> w$
  3. $I(x, y)$ is structured enough on scales $< w$ that auto-correlation has peak within the apodizing window.

- Vertical flux satisfies (Induc. eq.) $\neq$ advection

$$\frac{\partial B_z}{\partial t} + \mathbf{v}_h \cdot \nabla B_z \; = \; -\underbrace{B_z \nabla \cdot \mathbf{v}_h}_{1} \; + \; \underbrace{\nabla \cdot (v_z \mathbf{B}_h)}_{2}$$

  1: Convergence amplifies B_z — effect on cross-correlation?

  2: Vertical flow & horizontal field $\implies \Delta\mathbf{x}/\Delta t$ = feature (footpoint?) velocity unrelated to plasma motion $\mathbf{v}_h$. (Demoulin & Berger 2003).

"WHAT ABOUT OTHER TRACKING METHODS?"

- Kusano et al. (2003) & Welsch et al. (2004) correct LCT
- Apply LCT to *line-of-sight* m-grams — find spatially-smooth *horizontal* flow field $\bar{\mathbf{w}}_h(x, y)$:

$$\frac{\partial B_z}{\partial t} \overset{\dagger}{\widehat{=}} -\bar{\mathbf{w}}_h \cdot \nabla_h B_z \qquad \text{(Advect)}$$

- Equation (Advect) does not constraint component of $\bar{\mathbf{w}}_h \perp \nabla_h B_z$. LCT demands **spatial smoothness** permiting 2 components of $\bar{\mathbf{w}}_h$ to be found from a single relation.
- Project $\perp$ to $\mathbf{B}(x, y)$ — i.e. assume $\mathbf{v}_{\|} = 0$

$$\bar{\mathbf{v}} = \left[\mathrm{I} - \frac{\mathbf{BB}}{|\mathbf{B}|^2} \right] \cdot \bar{\mathbf{w}}_h$$

- Assume LCT approximates (i.e. $\nabla \cdot \bar{\mathbf{w}}_h = 0$)

$$\frac{\partial B_z}{\partial t} \simeq -\nabla_h \cdot (\bar{\mathbf{w}}_h B_z) \quad .$$

- May conclude:

$$\bar{\mathbf{w}}_h B_z = B_z \bar{\mathbf{v}}_h - \mathbf{B}_h \bar{v}_z + \nabla \psi' \times \hat{\mathbf{z}}$$

- Is (Induc. Eq.) satisfied on all spatial scales? *Probably not.*

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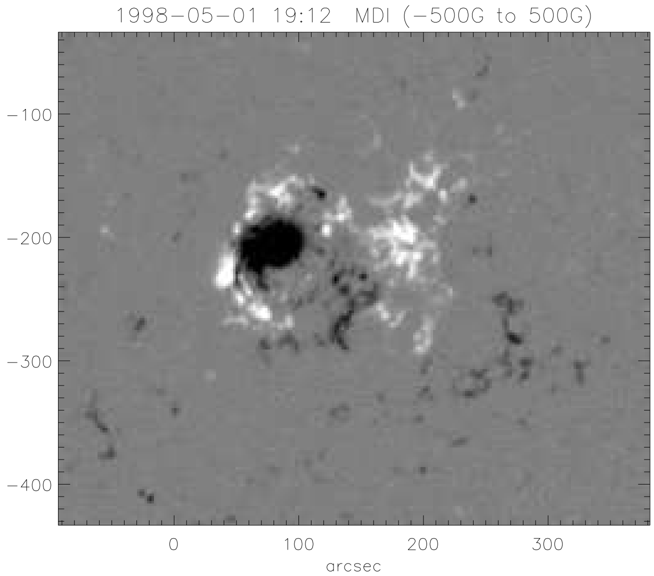
† This is an empirical relation. The LCT technique works according to localized cross-correlations — the velocity it returns is not guaranteed to satisfy any such equation.

“BUT DOES IT WORK ON REAL DATA?” AR 8210

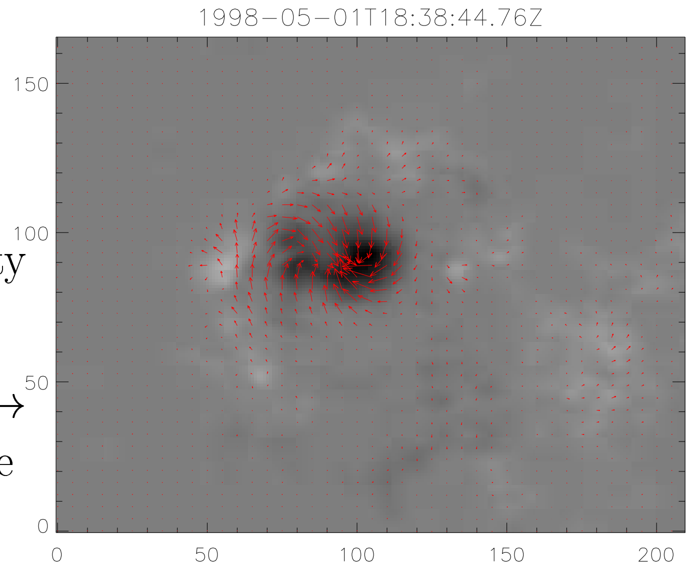
- 1998 May, 1
- Rotating neg’ve sunspot (Warmuth et al. 2000)
- EFR @ SE PIL (Sterling et al. 2001)
- GOES flares: 17:54 & 22:54 (Sterling et al. 2001)
- SE pos’ve conc’ns move South- Southwest-ward



IVM Vector M-grams

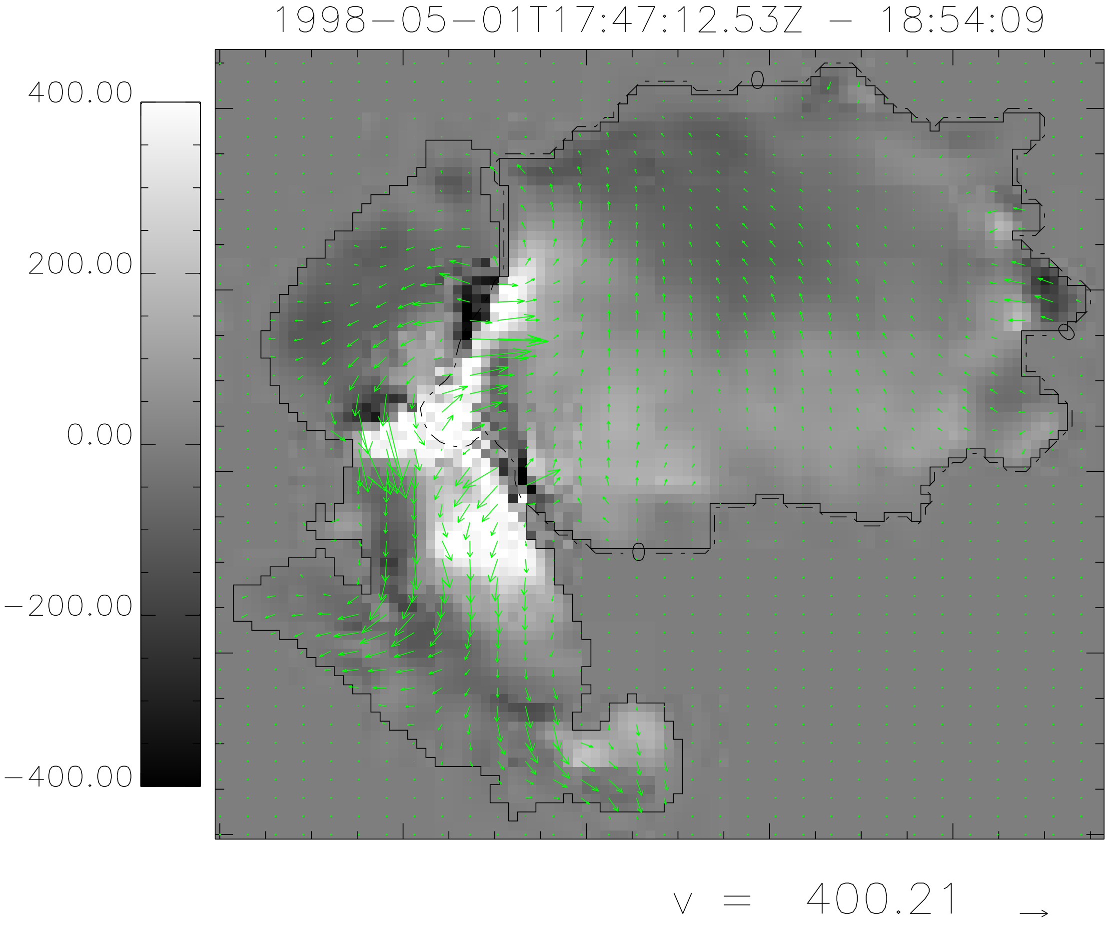
Mickey et al. 1996

- Cadence $\sim$ 3 minutes
- Co-aligned using intensity cross-correlation.
- 180° ambiguity resolved $\rightarrow$ least variable over sequence (Leka & Barnes 2003)



- Reduce noise by av’ging $\mathbf{B}(x, y)$ from 5 consecutive m-grams
- Select pair of avg’d m-grams from t_1 and $t_2 = t_1 + \Delta t$

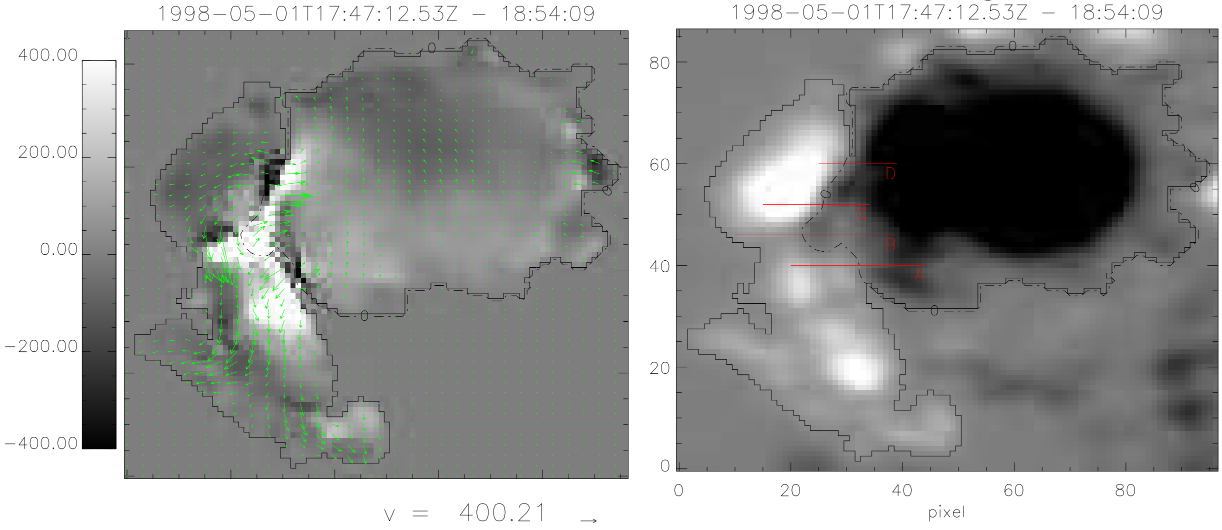
AR 8210 (cont'd)



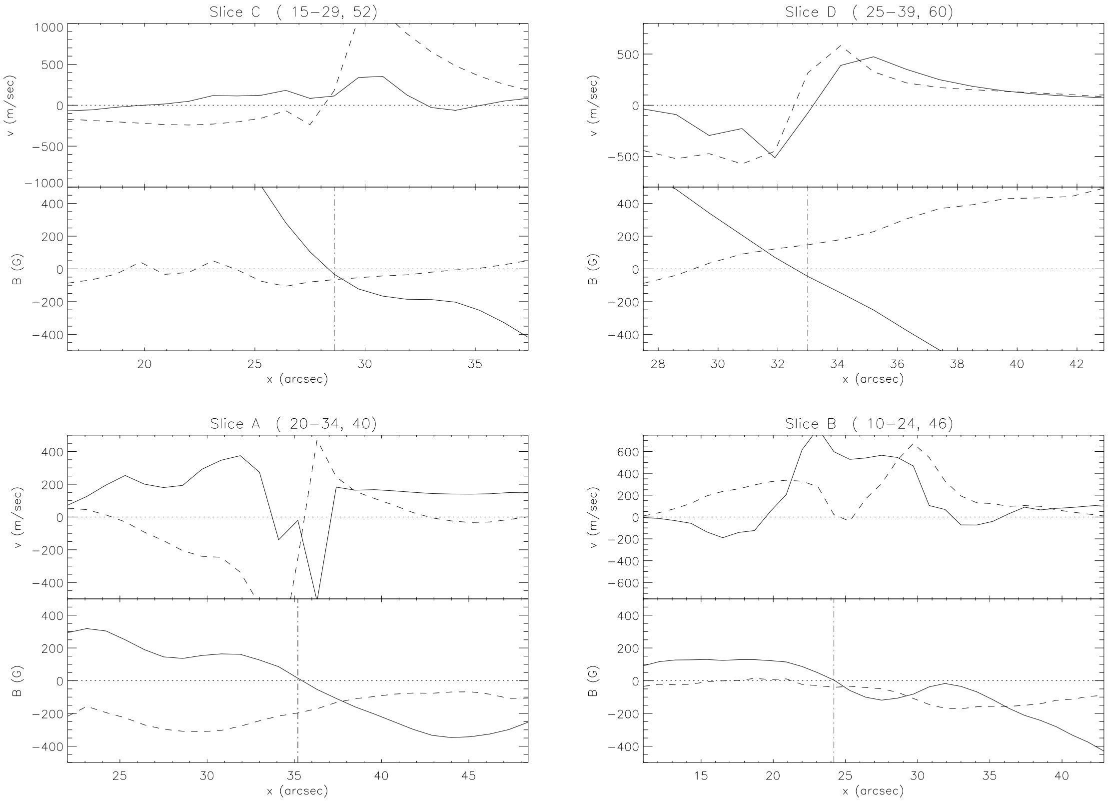
- $P = 1.5 \times 10^{27}\,\mathrm{erg/sec}$, $F = 6.5 \times 10^{22}\,\mathrm{cm^3/sec}$
- Positive region:
  - $\Phi = 2.3 \times 10^{21}\,\mathrm{Mx}$, $\dot{\Phi} = 2.1 \times 10^{16}\,\mathrm{Mx/sec}$
  - $F = 2.5 \times 10^{22}\,\mathrm{cm^3/sec}$, $\langle v_z \rangle = 25\,\mathrm{m/sec}$,
- Negative region:
  - $\Phi = 1.2 \times 10^{22}\,\mathrm{Mx}$, $\dot{\Phi} = 2.1 \times 10^{16}\,\mathrm{Mx/sec}$
  - $F = 3.7 \times 10^{22}\,\mathrm{cm^3/sec}$, $\langle v_z \rangle = 22\,\mathrm{m/sec}$,

AR 8210 (cont'd)

The structure of the PIL upflow — sampled along 4 slices:

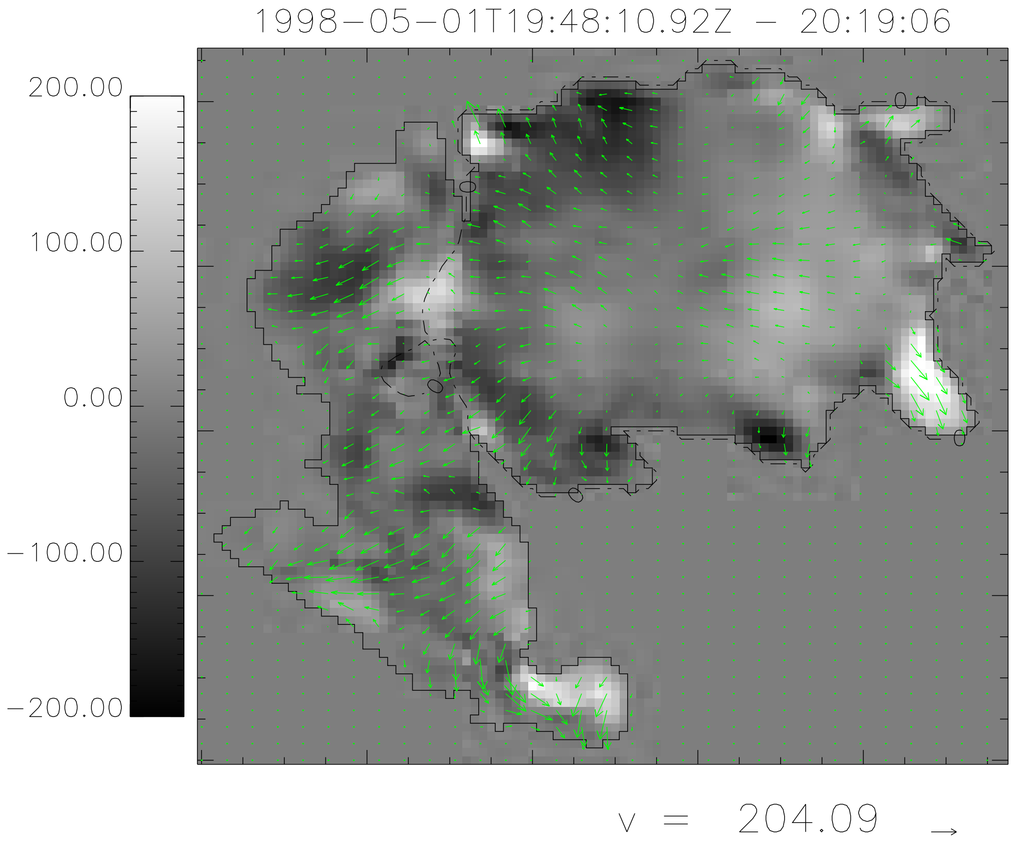


v_x, B_x (dashed) and v_z, B_z (solid) along slices



AR 8210 (cont'd)

Quieter period:



- $P = 1.9 \times 10^{27}$ erg/sec , $F = -3.0 \times 10^{22}\,\mathrm{cm}^3/\mathrm{sec}$
- Positive region:
  - $\Phi = 2.3 \times 10^{21}$ Mx, $\dot{\Phi} = -0.03 \times 10^{16}$ Mx/sec
  - $F = -1.4 \times 10^{22}\,\mathrm{cm}^3/\mathrm{sec}$, $\langle v_z \rangle = -15$ m/sec,
- Negative region:
  - $\Phi = 1.2 \times 10^{22}$ Mx, $\dot{\Phi} = -0.12 \times 10^{16}$ Mx/sec
  - $F = -1.4 \times 10^{22}\,\mathrm{cm}^3/\mathrm{sec}$, $\langle v_z \rangle = -9.5$ m/sec,
- $\Delta t = 31$ minutes, 1 pixel/Δt = 430 m/sec.

“HOW CAN I USE THE MEF?”

http://solar.physics.montana.edu/dana/mef/

```
;+
; NAME:    mef_vel
;
; PURPOSE:
;    Calculate the Minimum Energy Fit velocity field from a
;    a magnetogram pair.  This is the unique velocity field
;    producing a given change in normal field, which minimizes
;    the energy functional.
;
; CALLING SEQUENCE:
;
;    vel = mef_vel( mg, [ options ] )
;
; INPUTS:
;    mg: the magnetogram structure, containing
;        a single (averaged) vector magnetogram and its time derivative
;        on a regular grid with square pixels.
```

“WHAT’S NEXT?”

- More efficient implimentation of MEF elliptic solvers
  - Mutli-grid
  - Alternating-direction direct
- Test method on other data sets — perhaps from other vector magnetographs...
- Assess difficulties due to
  - 180° ambiguity resolution (esp. @ PIL)
  - Variable seeing
- Explored uses of “indep. information” field $\mathbf{u}$ from
  - $u_{\rm los}(x, y)$ from Stokes inversion
  - Sequence of inversions