LAST TIME (CH. 8)

- Capacitors
- Inductors
- Impedance

TODAY (CHS. 9-10)

- Impedance summary
- Transfer function (gain and phase)
- Examples
  - High pass filter (RC)
  - Resonance (LC)
ANNOUNCEMENT:

LAST TIME, I GOT THE SIGN WRONG IN THE INDUCTANCE EQUATION:

\[ V = L \frac{di}{dt} \] ← CORRECT

FAISSER HAS IT RIGHT. I HAVE FIXED MY LECTURE NOTES ACCORDINGLY.

ACTUALLY, I WAS ONLY FOLLOWING MY OWN UNDERGRADUATE TEXTBOOK! MOST PHYSICS (NOT ELECTRONICS) TEXTS HAVE A MINUS SIGN IN THE EQUATION. IT HAS TO DO WITH SIGN CONVENTIONS.

HERE IS THE SIGN CONVENTION WE USE.

PASSIVE COMPONENTS: \((L, R, C)\)

\[ \rightarrow \text{POSITIVE VOLTAGE} \]

\[ \Rightarrow \text{POSITIVE CURRENT} \]

POWER SOURCES:

\[ \rightarrow \text{POSITIVE VOLTAGE} \]

\[ \Rightarrow \text{POSITIVE CURRENT} \]

EXAMPLES:
IMPEDANCE SUMMARY

Inductors, Resistors & Capacitors can be modeled using impedance.

<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>VALUE</th>
<th>UNITS</th>
<th>DEFINING EQUATION</th>
<th>IMPEDANCE, Z (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>R</td>
<td>Ω</td>
<td>( V = i \cdot R )</td>
<td>( R )</td>
</tr>
<tr>
<td>Capacitor</td>
<td>C</td>
<td>F</td>
<td>( \dot{V} = CV \left( i = C \frac{dV}{dt} \right) )</td>
<td>( \frac{1}{jωC} )</td>
</tr>
<tr>
<td>Inductor</td>
<td>L</td>
<td>H</td>
<td>( V = L \frac{di}{dt} )</td>
<td>( jωL )</td>
</tr>
</tbody>
</table>

\[ V = V_0 \, e^{jωt}, \quad i = i_0 \, e^{jωt} \]

\[ V = \text{Re}(V), \quad i = \text{Re}(i) \]

Ohm's Law generalizes to:

\[ V = i \cdot Z \]

Frequency:

\[ f = \frac{ω}{2\pi} = \frac{1}{T} \]

Units: \( Hz \) (cycles per second)

Note: Impedances add just like resistance.

Series:

\[ Z_{TOT} = Z_1 + Z_2 + Z_3 \]

Parallel:

\[ Z_{TOT} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} \]
TRANSFER FUNCTIONS

Suppose some circuit modifies an input voltage:

\[ V_{\text{IN}} \xrightarrow{\text{SOME CIRCUIT}} V_{\text{OUT}} \]

Further suppose:

\[ V_{\text{OUT}} = G V_{\text{IN}} \]

Transfer function

Example:

\[ V_{\text{OUT}} = \frac{R_1 + R_2}{R_L} V_{\text{IN}} \]

\[ G = \frac{R_1 R_2}{R_L} \]

Example:

\[ V_{\text{OUT}} = \frac{R_2}{R_1 + R_2} V_{\text{IN}} \]

\[ G = \frac{R_2}{R_1 + R_2} \]

*Your text simply calls \( G \) the gain.
More generalized notion of a transfer function:

Let \( V_{in} = V_0 \ e^{i \omega t} \)
Assume \( V_{out} = G \ V_{in} \)

Now, if \( G \in \mathbb{C} \), then it has both a magnitude and a phase:

Let \( G = |G| \). This is the \underline{gain}.
\[ \frac{G}{G} = e^{i \phi} \] This is the \underline{phase factor}.

\[ V_{out} = G \ e^{i \phi} \ V_{in} = G \ V_0 \ e^{i(\omega t + \phi)} \]

Pictorially:
(Phasors)

Typically, \( G \) and \( \phi \) turn out to be functions of \( \omega \). This is why we call \( G = G(\omega) \) a transfer function.

* Faissler calls \( G \) the \underline{gain}. I think my terminology \((G = \text{transfer \, \, \, \, \, \, \, function}, \ |G| = \text{gain})\) is a little more standard, but I'll try to always be clear about what I'm asking for.
LAST TIME, WE ANALYZED AN RC LOWPASS FILTER:

\[ V_{in} \xrightarrow{R} V_{out} \]

\[ V_{out} = \frac{1 - j\omega RC}{1 + j\omega R^2 C^2} \cdot V_{in} \]

HERE IS A HIGH PASS FILTER:

\[ V_{in} \xrightarrow{C} V_{out} \]

USING NODE VOLTAGE ANALYSIS:

\[ \frac{V_{in} - V_{out}}{Z_C} + \frac{0 - V_{out}}{R} = 0 \Rightarrow R(V_{in} - V_{out}) - Z_C V_{out} = 0 \]

\[ \Rightarrow V_{out} = \frac{R}{R + Z_C} V_{in} \]

\[ G(\omega) = \frac{R}{Z_C + R} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{j\omega RC (1 - j\omega RC)}{(1 + j\omega RC)(1 - j\omega RC)} \]

\[ = \frac{\omega^2 R^2 C^2 + j\omega RC}{1 + \omega^2 R^2 C^2} \]

WHAT IS THE GAIN?

\[ G = \left| \frac{\omega^2 R^2 C^2 + j\omega RC}{1 + \omega^2 R^2 C^2} \right| = \frac{\sqrt{\omega^4 R^4 C^4 + \omega^2 R^2 C^2}}{1 + \omega^2 R^2 C^2} \]

\[ = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \]

\[ G \rightarrow \infty \quad \text{as} \quad \omega \rightarrow \infty \]

\[ G \rightarrow 0 \quad \text{as} \quad \omega \rightarrow 0 \]

\[ G \rightarrow 1 \quad \text{as} \quad \omega \rightarrow 1/RC \]

\[ \log \frac{G}{1} \quad \text{versus} \quad \omega \]

\[ G \rightarrow 1 \quad \text{as} \quad 1/RC \rightarrow \infty \]
WHAT ABOUT THE PHASE?

\[ e^{i\varphi} = \frac{G}{\bar{G}} = \frac{\frac{\omega^2 R^2 C^2 + j\omega R C}{1 + (\omega R C)^2}}{\omega R C} \cdot \frac{\sqrt{1 + (\omega R C)^2}}{\omega R C} = \frac{\omega R C + j}{\sqrt{1 + (\omega R C)^2}} = \cos \varphi + j \sin \varphi \]

so, \( \cos \varphi = \frac{\omega R C}{\sqrt{1 + (\omega R C)^2}} \), \( \sin \varphi = \frac{1}{\sqrt{1 + (\omega R C)^2}} \)

NOTE THAT \( \sin \varphi > 0 \) AND \( \cos \varphi > 0 \), SO THE PHASE \( \varphi \) IS IN QUADRANT I \( (0 < \varphi < \frac{\pi}{2}) \).

\[ \varphi = \sin^{-1} \left[ \frac{1}{\sqrt{1 + (\omega R C)^2}} \right] \]

\[ \begin{array}{c}
\varphi \\
\frac{\pi}{2} \\
\frac{\pi}{4} \\
0
\end{array} \quad \begin{array}{c}
\omega \\
\frac{1}{\sqrt{RC}}
\end{array} \quad \begin{array}{c}
\text{lag} \omega R C
\end{array} \]

Q: \( \varphi > 0 \) MEANS WHAT?
A: \( V_{out} \) LEADS \( V_{in} \).
What is the impedance of this combination?

\[ Z = Z_L + Z_C = j\omega L + \frac{1}{j\omega C} = j \left[ \omega L - \frac{1}{\omega C} \right] \]

\[ |Z| = \omega L - \frac{1}{\omega C} \]

Note that \( Z = 0 \) when \( \omega L = \frac{1}{\omega C} \), i.e., \( \omega = \frac{1}{\sqrt{LC}} \).

This condition is called **resonance**.

Investigating further, let's apply an A.C. voltage and consider the resulting voltage across the inductor:

\[ V_{IN} \quad \frac{1}{Z} \quad V_{OUT} \]

\[ V_{OUT} = \left( \frac{Z_L}{Z_L + Z_C} \right) V_{IN} \]

\[ G(\omega) = \frac{j\omega L}{j\omega L + \frac{1}{j\omega C}} = \frac{-\omega^2 LC}{-\omega^2 LC + 1} = \frac{\omega^2 LC}{\omega^2 LC - 1} \in \mathbb{R} \]

So \( G(\omega) = G(\omega) \); \( V_{OUT} = V_{IN} \), \( e^{\omega} = 1 \); \( \varphi = 0 \)

The voltage across the inductor is in phase with \( V_{IN} \).

\[ \begin{align*}
G(\omega) & \rightarrow 1 \text{ as } \omega \rightarrow \infty \\
& < 0 \text{ for } \omega < \frac{1}{\sqrt{LC}} \\
& > 1 \text{ for } \omega > \frac{1}{\sqrt{LC}}
\end{align*} \]
What happens in the neighborhood of $\omega \approx \frac{1}{\sqrt{LC}}$?

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{\sqrt{LC}} + \epsilon$</td>
<td>$G \gg 1$</td>
</tr>
<tr>
<td>$G \to \infty$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{LC}}$</td>
<td>Undefined! $</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{LC}} - \epsilon$</td>
<td>$G \ll -1$</td>
</tr>
<tr>
<td>$G \to -\infty$</td>
<td></td>
</tr>
</tbody>
</table>

In real life, if you try to tune to resonance (assuming nothing explodes!), then you learn that the circuit has a little resistance. This eliminates the bad mathematical behavior. It also introduces a gradual phase shift in the neighborhood of resonance.