

WE'VE BEEN ANALYZING AC CIRCUITS, USING COMPLEX IMPEDANCE, AND FINDING THE RESPONSE $G(\omega)$ (GAIN AND PHASE) FOR PASSIVE AND ACTIVE LINEAR FILTERS.

TODAY WE WILL PUT SOME MATHY MEAT ON THOSE BONES!

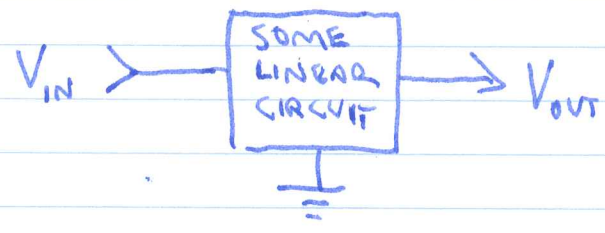
OUTLINE

- LINEARITY & SUPERPOSITION
- FOURIER SERIES
- POWER (AND RMS VOLTAGE)
- POWER SPECTRUM (AND dBV)

IF TIME PERMITS, I'LL COMMENT ON ALIASING AND WINDOW FUNCTIONS FOR THE FFT. SEE HANDOUTS ON THE WEB!

LINEARITY & SUPERPOSITION

RESISTORS, CAPACITORS, INDUCTORS, AND IDEAL OP AMPS WITH NEGATIVE FEEDBACK COMBINE TO FORM CIRCUITS WITH LINEAR EQUATIONS OF MOTION:



THE EQUATIONS OF MOTION HAVE THE FORM

$$D V_{OUT}(t) = V_{IN}(t),$$

WHERE D IS A LINEAR INTEGRO-DIFFERENTIAL OPERATOR. LINEAR OPERATORS HAVE THE PROPERTY

IF $D a = A$ AND $D b = B$,
 THEN $D (a+b) = A+B$.

CONSEQUENTLY, IF WE KNOW SOLUTIONS $V_{OUT,n}$ FOR SOME SET OF INPUTS $V_{IN,n}$ ($n=1,2,3,\dots$), THEN GIVEN A COMPOSITE INPUT

$$V_{IN}(t) = \sum_n \alpha_n V_{IN,n}(t) \quad (\alpha_n \in \mathbb{R} \text{ CONSTANTS})$$

THEN THE OUTPUT IS SIMPLY

$$V_{OUT}(t) = \sum_n \alpha_n V_{OUT,n}(t).$$

THIS IS CALLED THE PRINCIPLE OF SUPERPOSITION.

LINEAR EQUATIONS ARE COMMONLY ENCOUNTERED IN ALL AREAS OF PHYSICS (MAXWELL'S EQUATIONS AND SCHRÖDINGER'S EQUATION ARE IMPORTANT EXAMPLES, AS ARE THE EQUATIONS OF ELASTICITY GOVERNING SEISMIC WAVES).

FOURIER ANALYSIS

WE HAVE SO FAR ANALYZED AC CIRCUITS WITH A SIMPLE SINUSOIDAL INPUT: $\underline{V}_{IN} = A e^{j(\omega t + \phi)} = \underline{A} e^{j\omega t}$.

THE CIRCUIT RESPONSE IS DESCRIBED BY TRANSFER FUNCTION \underline{G} :

$$\underline{V}_{OUT} = \underline{G}(\omega) \underline{V}_{IN}.$$

SUPPOSE WE HAVE INPUTS AT MULTIPLE FREQUENCIES ω_n :

$$\underline{V}_{IN}(t) = \sum_{n=0}^{\infty} A_n e^{j(\omega_n t + \phi_n)} \quad (1)$$

SINCE THE CIRCUIT IS LINEAR, THE OUTPUT IS DETERMINED BY SUPERPOSITION:

$$\underline{V}_{OUT}(t) = \sum_{n=0}^{\infty} \underline{G}(\omega_n) A_n e^{j(\omega_n t + \phi_n)} \quad (2)$$

IT TURNS OUT THAT ANY PERIODIC SIGNAL CAN BE DESCRIBED IN THE FORM GIVEN BY (1). SUPPOSE THE PERIOD IS T . THEN LET $\omega_n = 2\pi n/T$.

$n = 0$ CORRESPONDS TO "DC OFFSET," THE MEAN VALUE OF $V(t)$.

$n = 1$ HAS PERIOD T , CALLED THE "FUNDAMENTAL."

$n \geq 2$ ARE "HARMONICS."

EQUATION (1) IS ONE FORM OF THE FOURIER SERIES FOR \underline{V}_{IN} .

EVERYTHING ABOUT THE INPUT SIGNAL IS ENCODED IN THE COMPLEX COEFFICIENTS

$$\underline{A}_n = A_n e^{j\phi_n}, \quad n = 0, 1, 2, \dots$$

LIKEWISE THE OUTPUT VOLTAGE IS DESCRIBED BY COEFFICIENTS

$$\underline{A}'_n = \underline{A}_n \underline{G}(\omega_n).$$

I WANT TO CAST THE FOURIER SERIES (1) IN A MORE USEFUL FORM. BUT FIRST, I NEED TO EXPLAIN SOMETHING ABOUT FUNCTIONS AND THEIR SYMMETRIES.

EVEN & ODD FUNCTIONS

AN EVEN FUNCTION IS SYMMETRIC ABOUT $t=0$. THAT IS, $f(t) = f(-t)$. $\cos \omega t$ IS AN EXAMPLE.

AN ODD FUNCTION IS ANTISYMMETRIC ABOUT $t=0$. THAT IS, $f(t) = -f(-t)$. $\sin \omega t$ IS AN EXAMPLE.

SUPPOSE NOW THAT $f(t)$ IS NEITHER EVEN NOR ODD. CONSIDER

$$g(t) = \frac{1}{2}(f(t) + f(-t)) \quad \leftarrow g \text{ IS EVEN}$$

$$h(t) = \frac{1}{2}(f(t) - f(-t)) \quad \leftarrow h \text{ IS ODD}$$

$$\text{BUT } f = g + h,$$

SO ANY FUNCTION CAN BE WRITTEN AS A SUM OF EVEN AND ODD PARTS.



EXAMPLE: POLYNOMIAL $P(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots$

CAN BE WRITTEN AS $P(x) = P_{\text{EV}}(x) + P_{\text{ODD}}(x)$ WHERE

$$P_{\text{EV}}(x) = C_0 + C_2x^2 + \dots \quad (\text{EVEN})$$

$$P_{\text{ODD}}(x) = C_1x + C_3x^3 + \dots \quad (\text{ODD}).$$

FOURIER SINE & COSINE SERIES

NOW RETURN TO EQ. (1). TAKING THE REAL (PHYSICAL) PART,

$$V(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t + \phi_n), \quad \omega_n = \frac{2\pi n}{T} = 2\pi f_n.$$

FOR EVEN FUNCTIONS, $\phi_n = 0$ AND

$$V_{\text{EVEN}}(t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t) = D + \sum_{n=1}^{\infty} a_n \cos(\omega_n t)$$

↑
DC OFFSET (FORMERLY a_0)

FOR ODD FUNCTIONS, LET $\phi_n = -\frac{\pi}{2}$.

$$V_{\text{ODD}}(t) = \sum_{n=1}^{\infty} b_n \cos(\omega_n t - \frac{\pi}{2}) = \sum_{n=1}^{\infty} b_n \sin(\omega_n t)$$

SO IN GENERAL, ANY PERIODIC $V(t)$ CAN BE REPRESENTED:

$$V(t) = D + \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)]$$

WHERE $\omega_n = \frac{2\pi n}{T} = 2\pi f_n$

n=0 TERM ALWAYS ZERO.

SUPPOSE WE MULTIPLY BOTH SIDES BY $\cos(\omega_m t)$ AND INTEGRATE:

$$\int_{-T/2}^{T/2} V(t) \cos(\omega_m t) dt = \int_{-T/2}^{T/2} \left\{ D + \sum_n [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \right\} \cos(\omega_m t) dt = \frac{a_m T}{2}.$$

ALL TERMS IN $\{\}$ EXCEPT $a_m \cos^2(\omega_m t)$ INTEGRATE TO ZERO.*
(NOT HARD TO SHOW, BUT TRUST ME FOR NOW...) SO THAT

$$a_m = \frac{2}{T} \int_{-T/2}^{T/2} V(t) \cos(\omega_m t) dt.$$

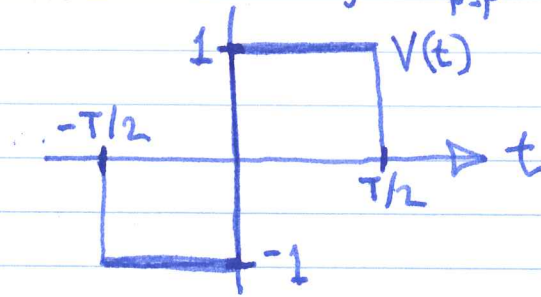
SIMILARLY,

$$b_m = \frac{2}{T} \int_{-T/2}^{T/2} V(t) \sin(\omega_m t) dt.$$
$$D = \frac{1}{T} \int_{-T/2}^{T/2} V(t) dt. \quad (\text{DC OFFSET})$$

$a_m, b_m,$ AND D
ARE CALLED FOURIER
COEFFICIENTS.

*THIS IS SOMETIMES CALLED "FOURIER'S TRICK".

EXAMPLE — SQUARE WAVE, $V_{p-p} = 2$



AS DRAWN, THIS IS AN ODD FUNCTION, SO $D = a_n = 0$.

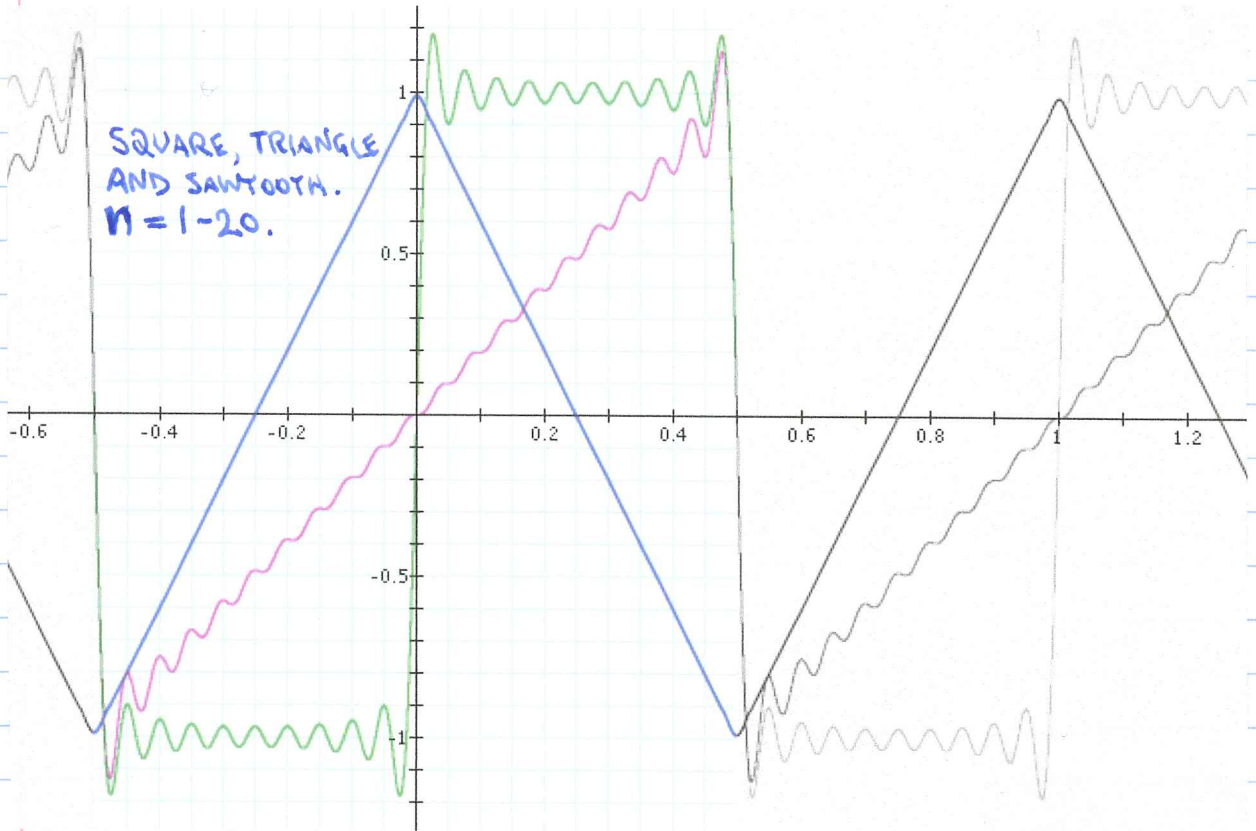
$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} V(t) \sin(2\pi n t / T) dt$$

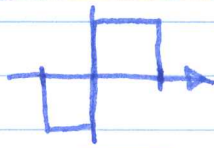
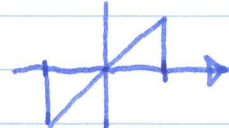
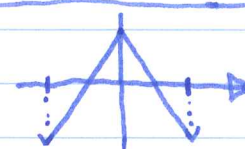
$$= \frac{4}{T} \int_0^{T/2} \sin(2\pi n t / T) dt$$

$$= -\frac{4}{2\pi n} [\cos(\pi n) - 1]$$

$$b_n = \begin{cases} \frac{4}{\pi n}, & n \text{ ODD} \\ 0, & n \text{ EVEN} \end{cases}$$

SOME EXAMPLES ($D=0$).



$V(t)$	a_n	b_n
SINUSOID $\cos(2\pi t/T - \phi)$	$a_1 = \cos\phi$ $a_{2+} = 0$	$b_1 = \sin\phi$ $b_{2+} = 0$
SQUARE 	0	n ODD: $4/\pi n$ n EVEN: 0
SAWTOOTH 	0	$\frac{2}{\pi n} (-1)^{n+1}$
TRIANGLE 	n ODD: $\frac{8}{\pi^2 n^2}$ n EVEN: 0	0

POWER (AND RMS VOLTAGE)

SUPPOSE $V = V_0 \cos(\omega t + \phi)$.

IF WE DO NOT KNOW THE LOAD RESISTANCE, ALL WE CAN DO IS SUPPOSE THAT, SINCE $P = IV = V^2/R$,

$$P \propto V^2$$

SO, THE AVERAGE POWER FOR UNIT RESISTANCE IS:

$$\begin{aligned} \bar{P} &= \overline{V^2} = \frac{1}{T} \int_{-T/2}^{T/2} V^2 dt \\ &= \frac{V_0^2}{T} \int_{-T/2}^{T/2} \cos^2(\omega t + \phi) dt = \frac{V_0^2}{2} \end{aligned}$$

THE ROOT MEAN SQUARE (RMS) VOLTAGE IS

$$V_{\text{RMS}} = \sqrt{\overline{V^2}} = \frac{V_0}{\sqrt{2}}$$

$$\text{AND } \bar{P} = \frac{V_{\text{RMS}}^2}{R} \quad \text{--- JUST LIKE THE INSTANTANEOUS POWER FORMULA.}$$

POWER SPECTRUM

WHAT HAPPENS WHEN WE CALCULATE $\overline{V^2}$ FOR

$$V(t) = D + \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] ?$$

$$\overline{V^2} = \frac{1}{T} \int_{-T/2}^{T/2} V^2 dt$$

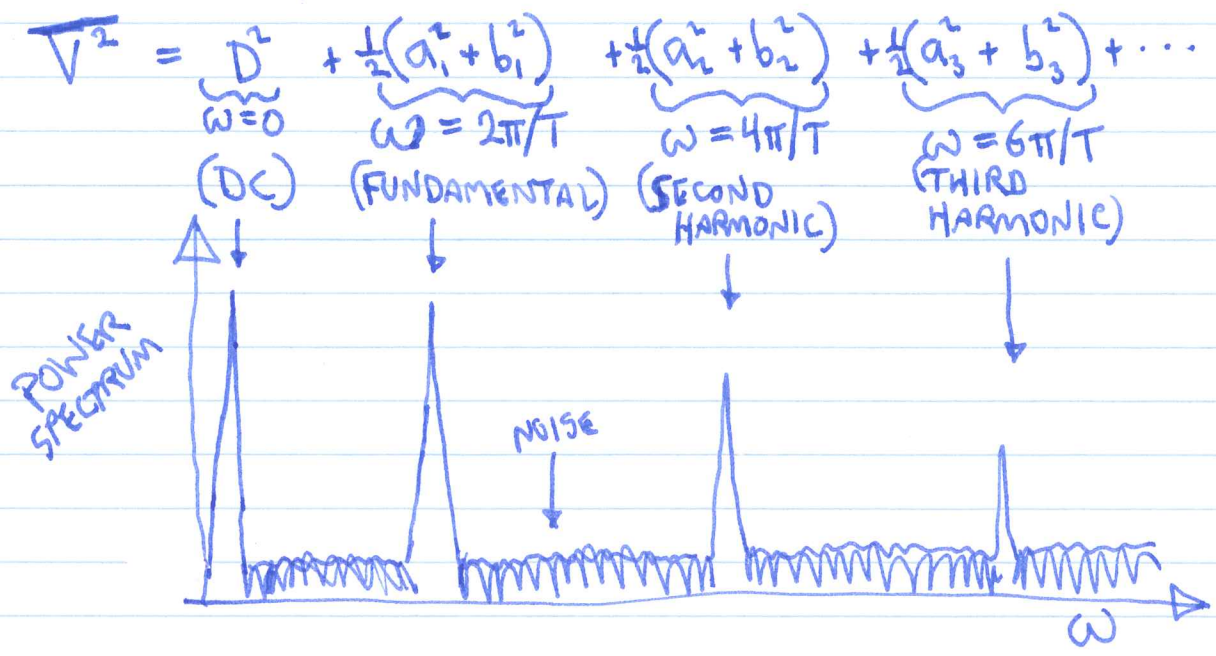
WHEN WE SQUARE AND INTEGRATE, ALL THE CROSS TERMS VANISH, JUST LIKE WHAT HAPPENED IN "FOURIER'S TRICK" A FEW PAGES AGO. SO:

$$\overline{V^2} = \frac{1}{T} \int_{-T/2}^{T/2} \left\{ D^2 + \sum [a_n^2 \cos^2(\omega_n t) + b_n^2 \sin^2(\omega_n t)] \right\} dt$$

$$\overline{V^2} = D^2 + \frac{1}{2} \sum_n (a_n^2 + b_n^2)$$

$$V_{RMS} = \sqrt{D^2 + \frac{1}{2} \sum_n (a_n^2 + b_n^2)}$$

THE POWER SPECTRUM IS A SERIES OF PEAKS AS A FUNCTION OF ω :



dBV IS THE USUAL UNIT FOR POWER SPECTRA.

20 dBV	= 10 V _{RMS} ,	$\overline{V^2} = 100 \text{ V}^2$	} INDEPENDENT OF LOAD!
0 dBV	= 1 V _{RMS} ,	$\overline{V^2} = 1 \text{ V}^2$	
-20 dBV	= 0.1 V _{RMS} ,	$\overline{V^2} = 0.01 \text{ V}^2$	

EXAMPLE:

$$\begin{aligned}
 V_1(t) &= 5V (1 + \sin(\omega t)) \\
 &= \frac{5V}{D} + \frac{5V}{b_1} \sin(\omega t) \\
 \overline{V_1^2} &= (5V)^2 + \frac{1}{2}(5V)^2 \\
 &= 25 \text{ V}^2 + 12.5 \text{ V}^2 = 37.5 \text{ V}^2 *
 \end{aligned}$$

WHAT IS THE MEAN POWER IF THE LOAD IS 10Ω?

$$\overline{P} = \frac{\overline{V_1^2}}{R} = 3.75 \text{ W} .$$

WHAT IS THIS IN dBV? (TRICK QUESTION)

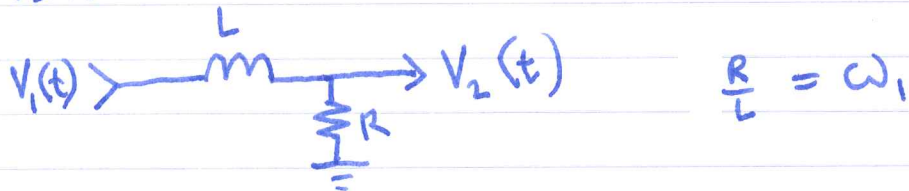
WE KNOW THAT dBV IS INDEPENDENT OF LOAD — SO LEAVE OUT R!

$$\begin{aligned}
 P(\text{dBV}) &= 10 \log_{10} (\overline{V_1^2}) = 10 \log_{10} (37.5) \\
 &= 15.7 \text{ dBV}
 \end{aligned}$$

*EACH TERM IN THE FOURIER SERIES HAS A POWER. THE POWERS (NOT IN dBV) ADD LINEARLY.

EXAMPLE

SUPPOSE $V_1(t)$ ON THE PREVIOUS PAGE IS PUT INTO A FILTER:



WHAT IS THE OUTPUT SPECTRUM?

$$\left| \frac{\tilde{V}_2}{\tilde{V}_1} \right| = G(\omega) = \left| \frac{R}{R + j\omega L} \right| = \left| \frac{1}{1 + j\omega/\omega_1} \right|$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_1}\right)^2}} ; G_{dB} = 20 \log_{10} \left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_1}\right)^2}} \right) \\ = -10 \log_{10} \left(1 + \left(\frac{\omega}{\omega_1}\right)^2 \right)$$

ω	$\overline{V_1^2}$	$G(\omega)$	$\overline{V_2^2}$
0	25 V ² 14.0 dBV	1 0 dB	25 V ² 14.0 dBV
ω_1	12.5 V ² 11.0 dBV	$1/\sqrt{2}$ -3 dB	6.25 V ² 8.0 dBV
TOTAL POWER	37.5 V ² 15.7 dBV		31.3 V ² 15 dBV