

Today

2/15/11 Lecture 5

Fourier Series

- Time-Frequency Decomposition/Superposition
- Fourier Components (Ex. Square wave)
- Filtering
- Spectrum Analysis
 - Windowing
 - Fast Fourier Transform
 - Sweep Frequency Analyzer

Homework: (due next Tuesday)

- 1) Write down the expected powers and dBVs for the 3rd harmonic of all four functions in the lab if they were 2 Vpp functions (versus 1Vpp functions).
- 2) For a square wave of period 300 microseconds that goes from -1 volts to +1 volts into 50 ohms, what are the frequencies and powers in the 4 strongest frequency components? Does it matter how square wave is centered in time (i.e. odd or even with respect to $t=0$)?
- 3) How much power in watts is dissipated into a 50 ohm resistor by a -13dBV signal?
- 4) What is the ratio of the powers and the voltages of a -27dBV signal and a -33dBV signal?

Reading

- See Prelab
- Horowitz and Hill 2nd Ed., pages 1025-1038.
- Optional: see references at end of lecture.

Lab

Fourier Analysis

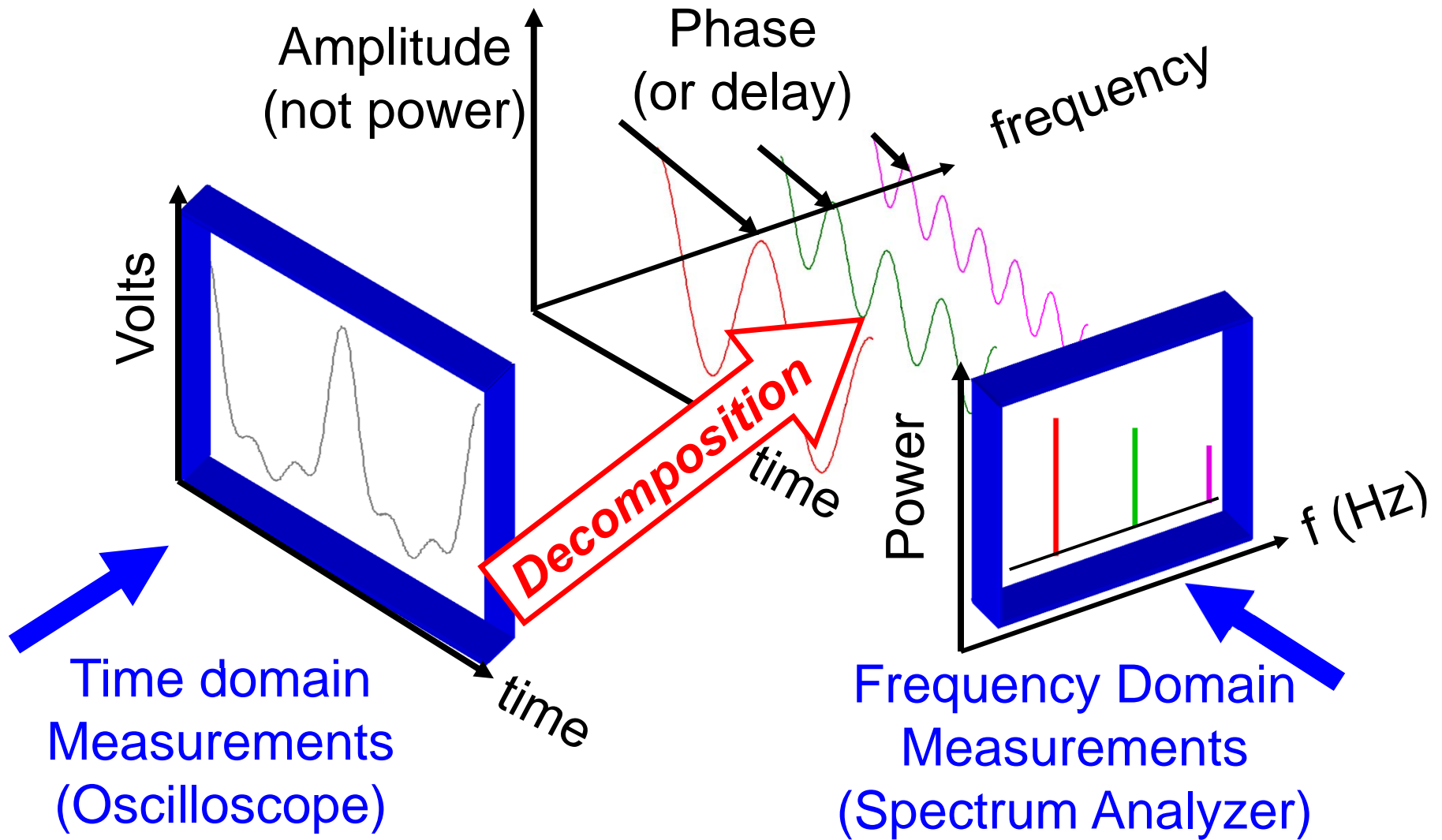
Do prelab before lab starts.

Fourier's Theorem

French mathematician Joseph Fourier (1768-1830), discovered that he could represent any real functions with a series of weighted sines and cosines.

In circuit analysis we use Fourier's Theorem to “decompose” a complex time domain signal into its discrete sinusoidal parts (the frequency domain.) Superposition of these frequency component returns the signal to the time domain.

The Time and Frequency Domains



Sine Wave in Time Domain

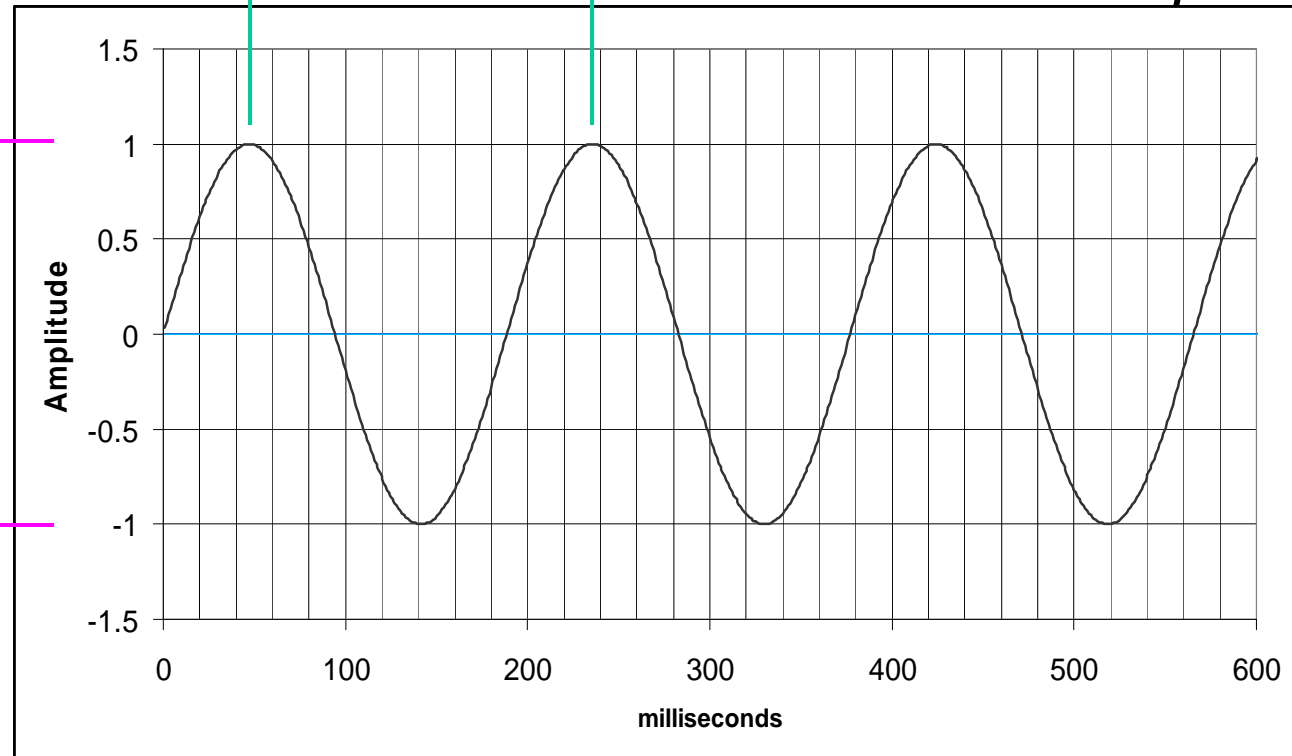
$$V(t) = A \sin 2\pi f t$$

$f = \text{frequency} = \frac{1}{\text{period}}$

$$A = \frac{V_{PP}}{2}$$

V_{pp}

$$V_{rms} = \sqrt{\langle |V(t)|^2 \rangle}$$

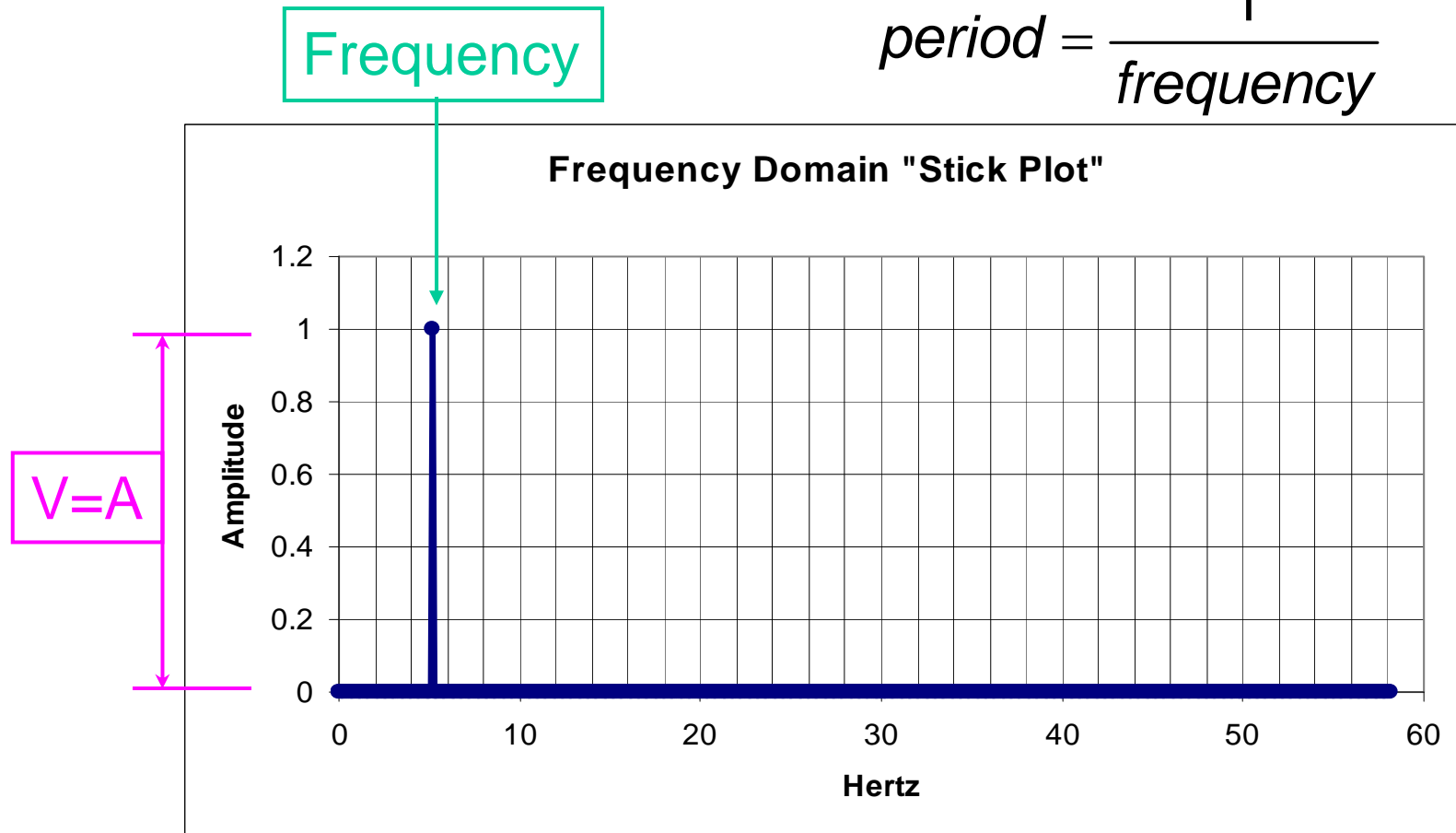


$$P = \text{Power} = \left\langle \frac{|V(t)|^2}{R} \right\rangle = \frac{V_{rms}^2}{R} = \frac{A^2}{2R}$$

For sine wave only

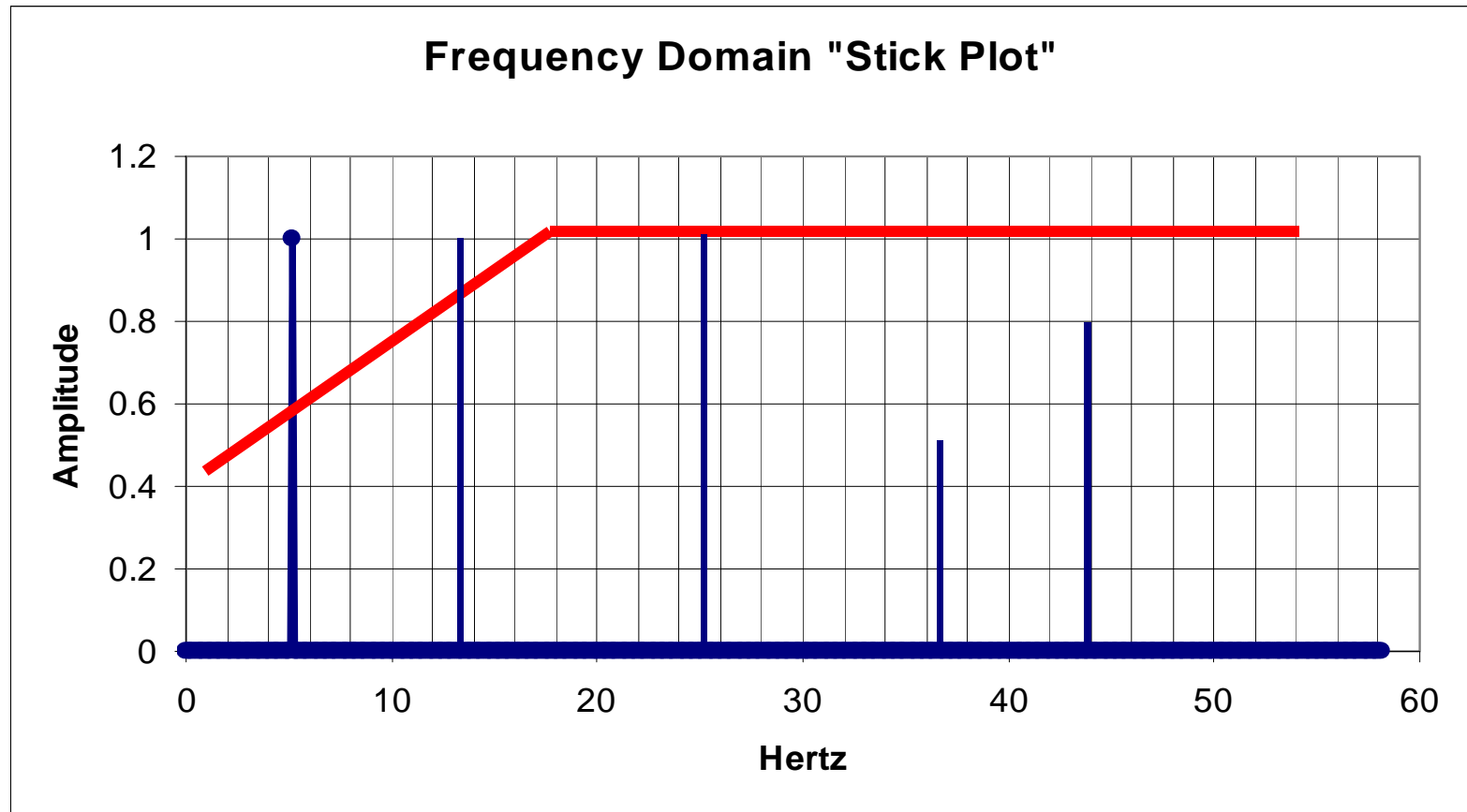
Sine Wave in Frequency Domain

$$period = \frac{1}{frequency}$$



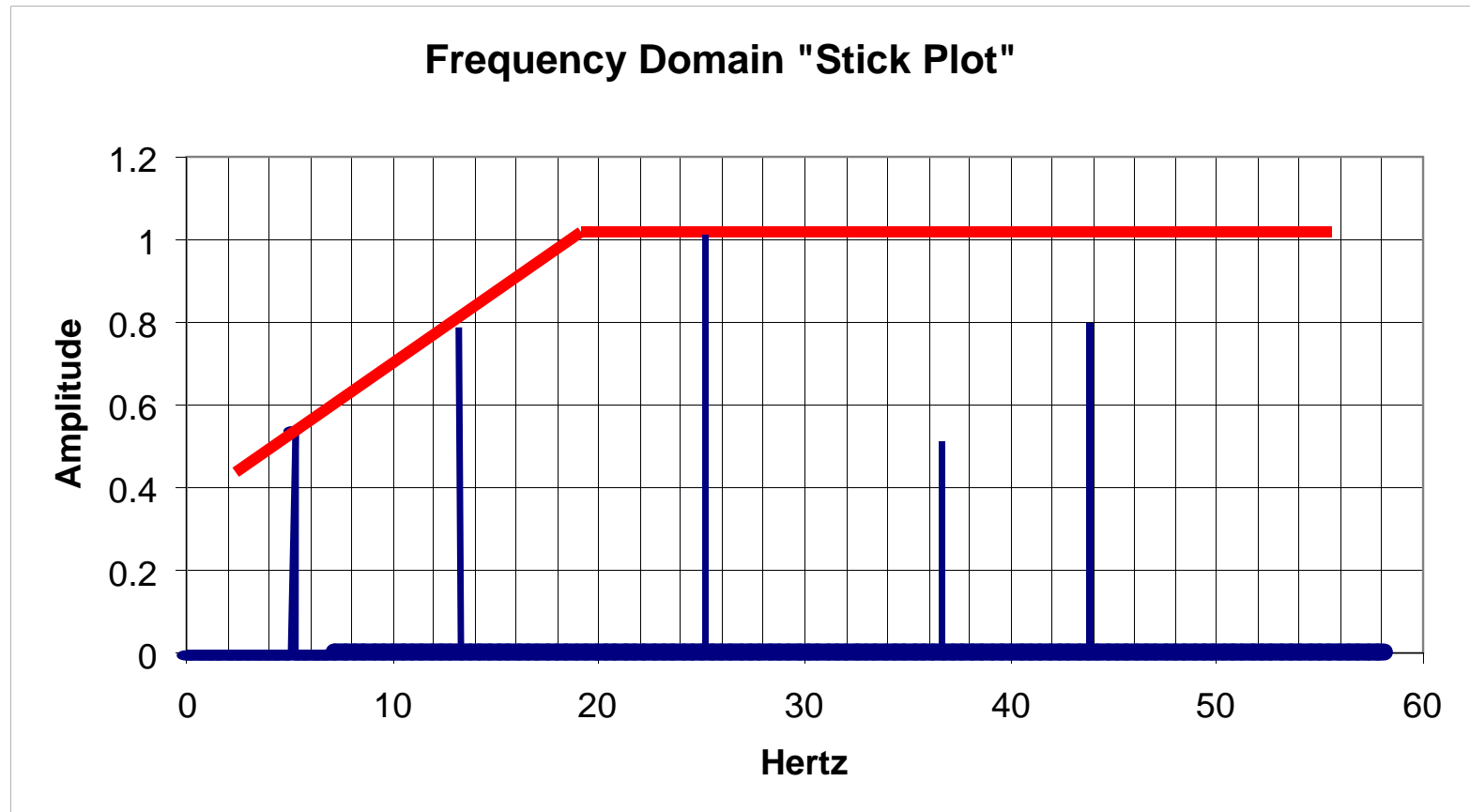
Amplitude-Spectrum Plot

Fourier Domain and Filtering



Amplitude-Spectrum Plot Overlaid by Gain-Frequency

Filtered Signal



Each component is transmitted at its filtered amplitude. Filter can also introduce phase shift of each component. Resultant signal is the sum of the transmitted components.

Fourier Series

(for periodic functions)

$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

⏟

DC

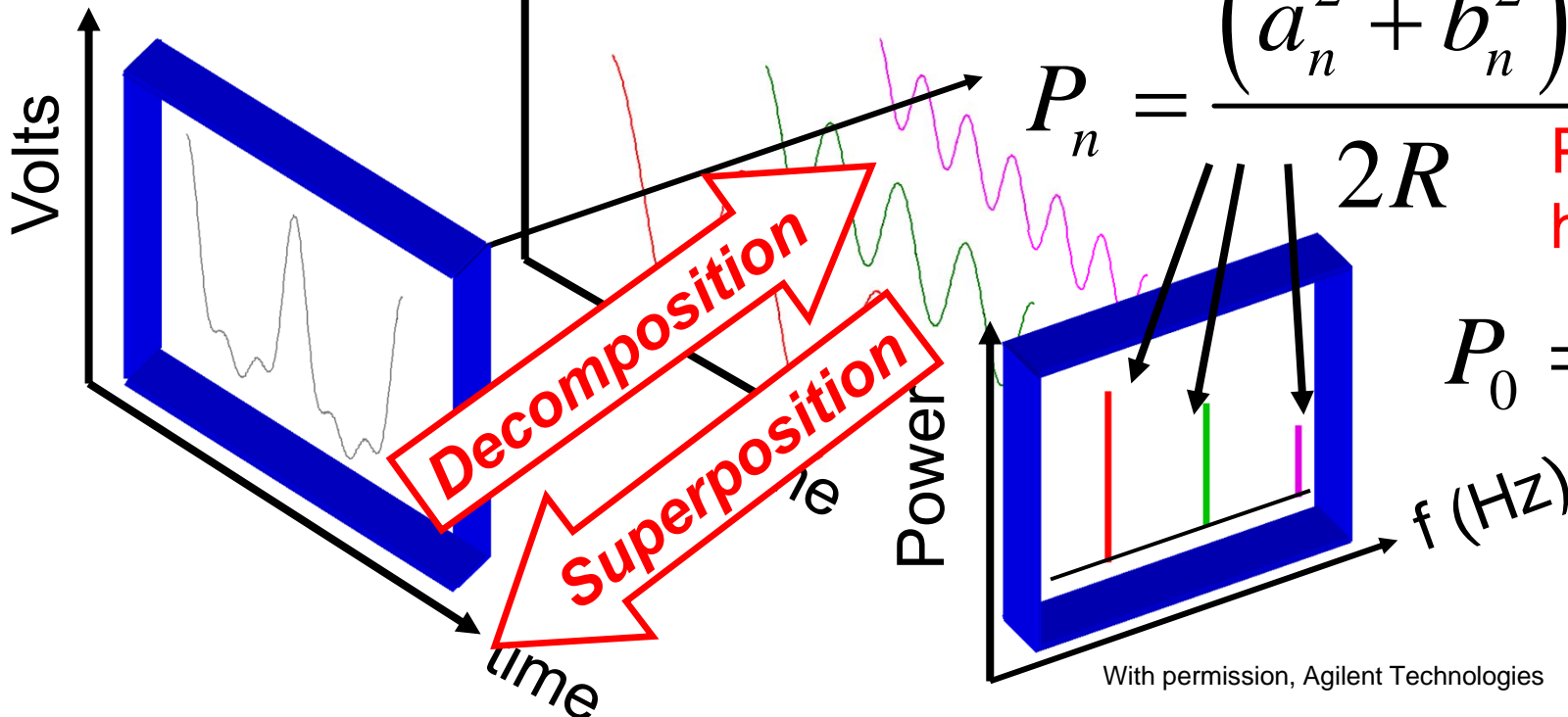
⏟

AC

$$P_n = \frac{(a_n^2 + b_n^2)}{2R}$$

Power in harmonics

$$P_0 = a_0^2 / R$$



With permission, Agilent Technologies

dBV

Since scope only measures voltage and doesn't know what load resistor you are using, it can't measure power absolutely, so if measure in dBV

dBV is a measure of relative POWER (not voltage)!!!!

A 20 dBV sinewave has 100 times more power than 0dBV and 10 times the voltage.

dBV is relative to the power of a sinewave relative to a 1 Volt RMS sinewave signal.

$$\text{dBV} = 10 \log_{10}(\langle V^2 \rangle / 1V_{\text{rms}}) = 10 \log_{10}(A^2/2)$$
where A is the amplitude of the sinewave in volts

Note dBV is it independent of resistive load.

Fourier Transform (Decomposition)

Fourier series:
$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$T_0 = \frac{2\pi}{\omega_0} = \textit{period}$$

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) dt$$

DC

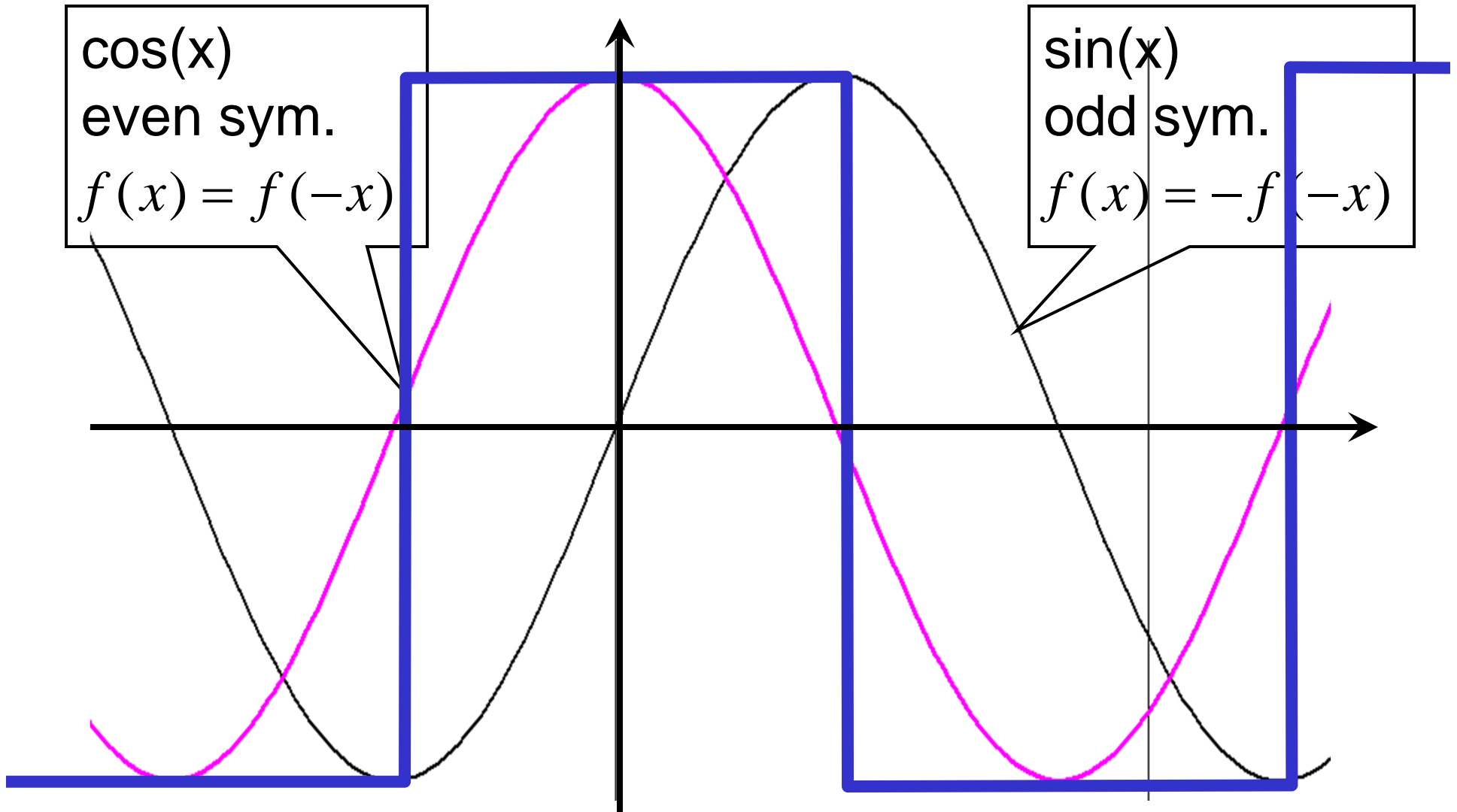
$$a_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \cos(2\pi n t / T_0) dt$$

Even
part of
V(t)

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \sin(2\pi n t / T_0) dt$$

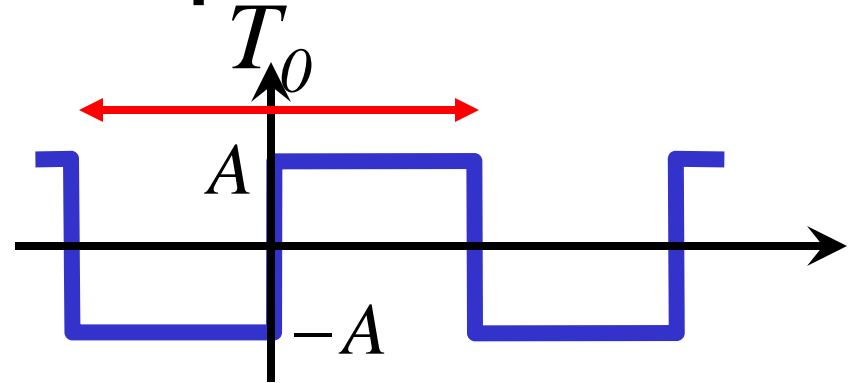
Odd
part of
V(t)

Odd and Even Symmetry



Fourier's a_0 for a Square Wave

$$V(t) = \begin{cases} A & 0 < t < T_0/2 \\ -A & -T_0/2 < t < 0 \end{cases}$$



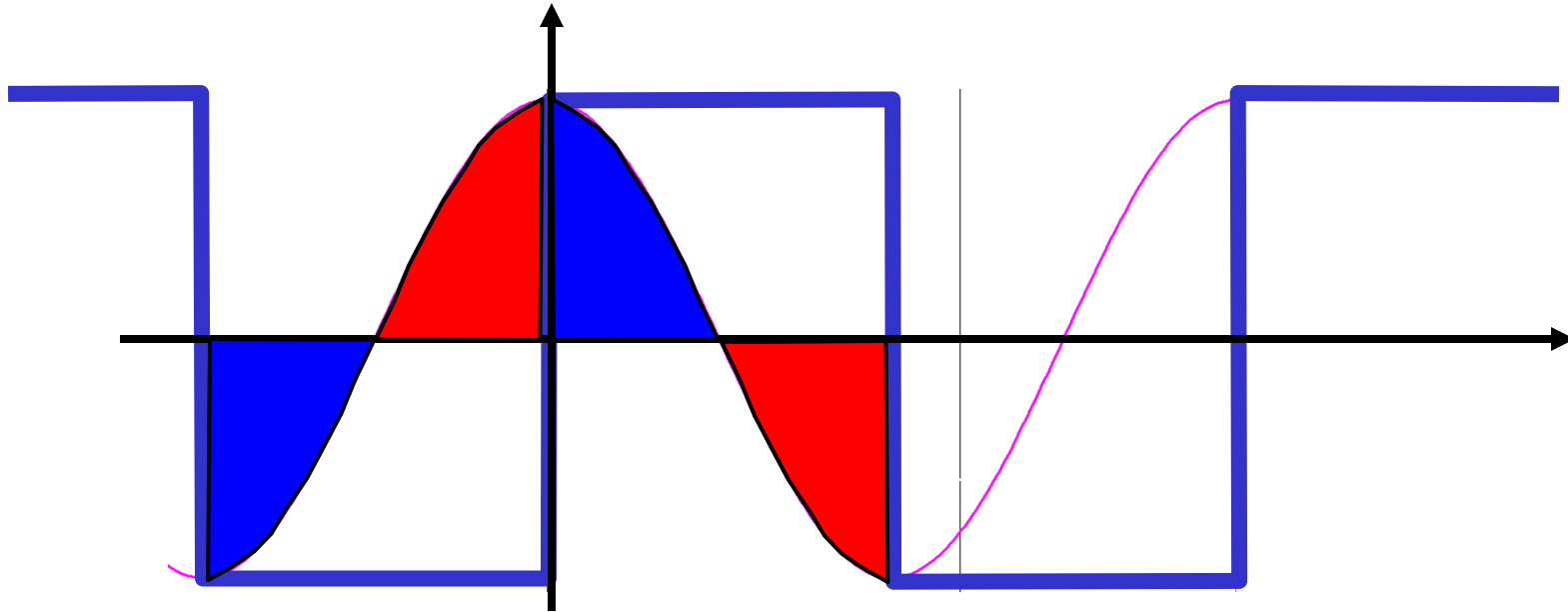
$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) dt = \frac{1}{T_0} \int_0^{+T_0/2} A dt + \frac{1}{T_0} \int_{-T_0/2}^0 (-A) dt$$

$$= \frac{A}{T_0} \left[\frac{T_0}{2} - \frac{T_0}{2} \right] = 0$$

For this waveform,
DC component is
precisely zero

Fourier's a_n for a Square Wave

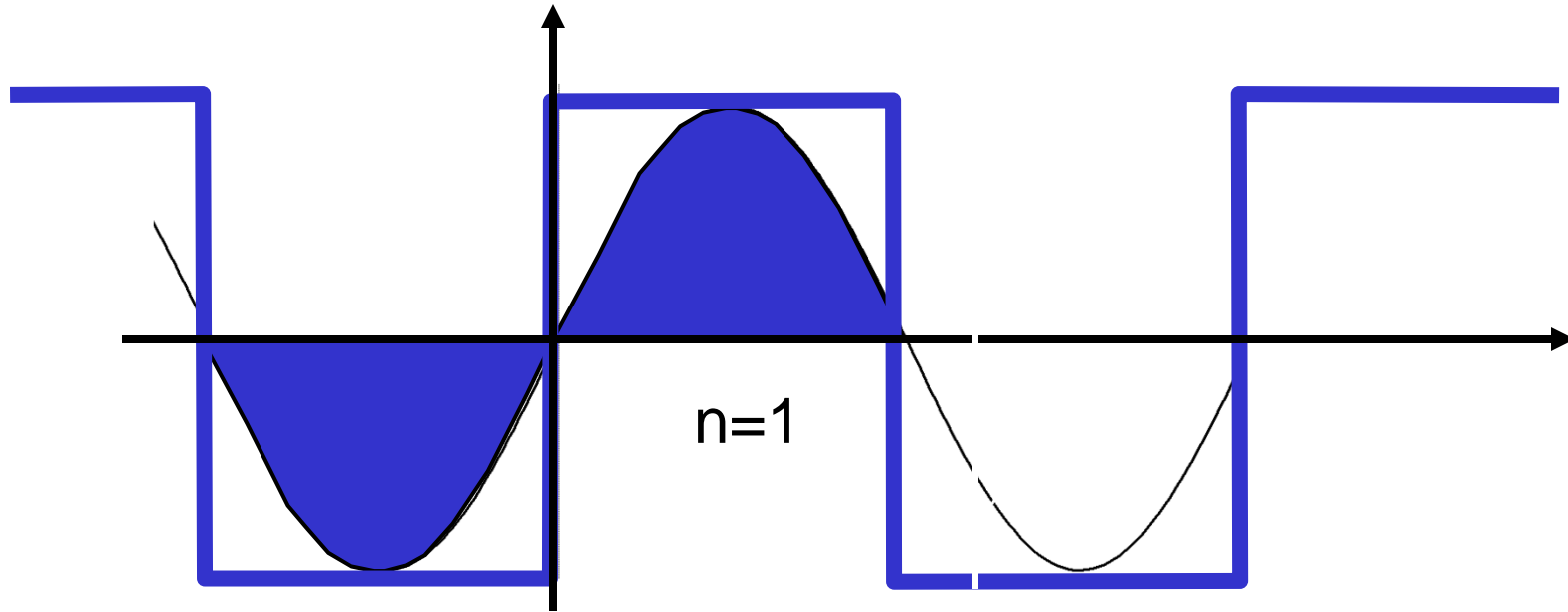
$$a_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \cos(2\pi nt/T_0) dt$$



$V(t)$ is odd $\therefore a_n = 0$ for all n

Fourier's b_n for a Square Wave

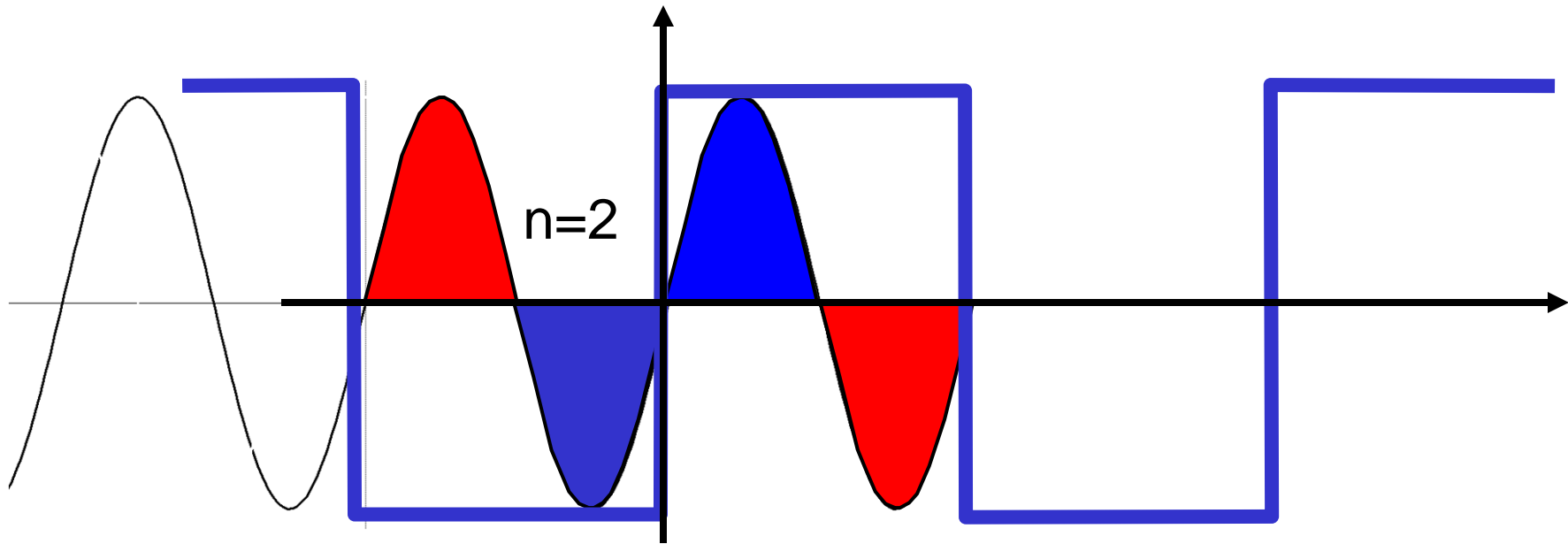
$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \sin(2\pi nt/T_0) dt$$



$$b_1 = \frac{4A}{\pi}$$

Fourier b_2

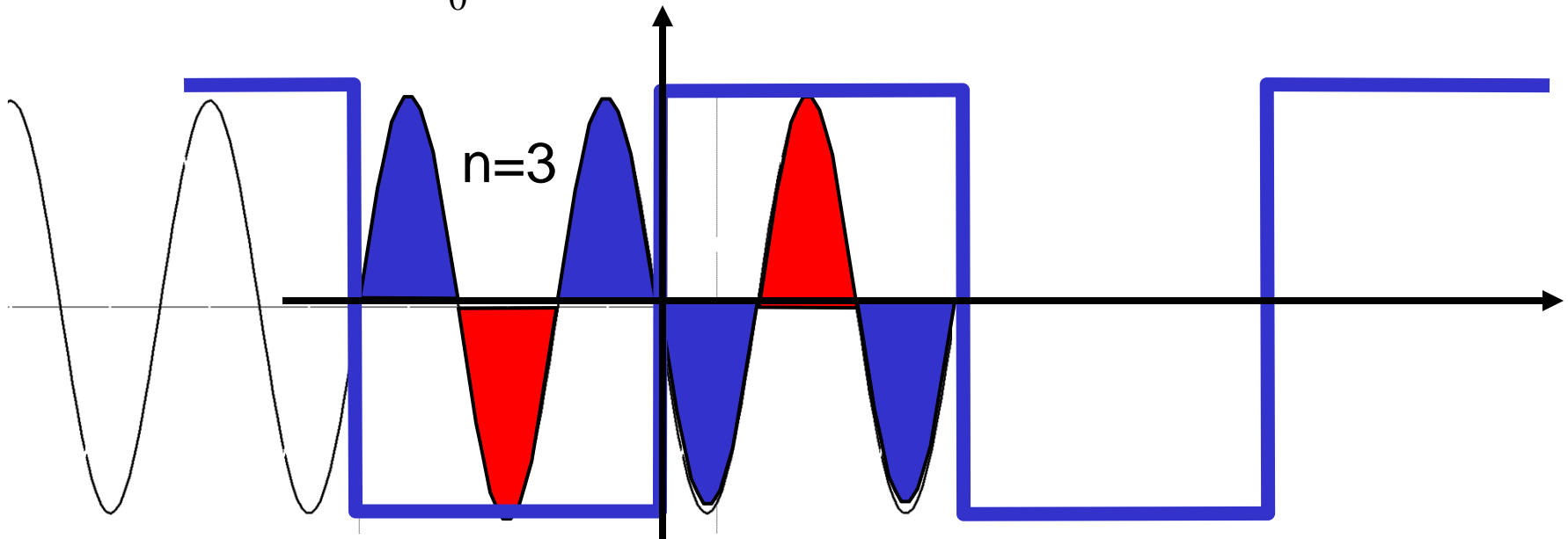
$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \sin(2\pi nt/T_0) dt$$



$$b_2 = 0$$

Fourier b_3

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} V(t) \sin(2\pi nt/T_0) dt$$



$$b_3 = \frac{4A}{3\pi}$$

$$b_3 = \frac{b_1}{3}$$

Fourier Series for Square Wave

Fourier's infinite series:

$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

For a square wave centered around ground and time=0:

$$a_0 = 0; \quad a_n = 0 \quad (\text{odd function});$$

$$b_n = \frac{4A}{n\pi} \quad (n \text{ odd}) \quad b_n = 0 \quad (n \text{ even})$$

$$V(t) = \frac{4A}{\pi} \left(\sin(\omega_0 t) + \frac{1}{3} \sin(3\omega_0 t) + \frac{1}{5} \sin(5\omega_0 t) + \dots \right)$$

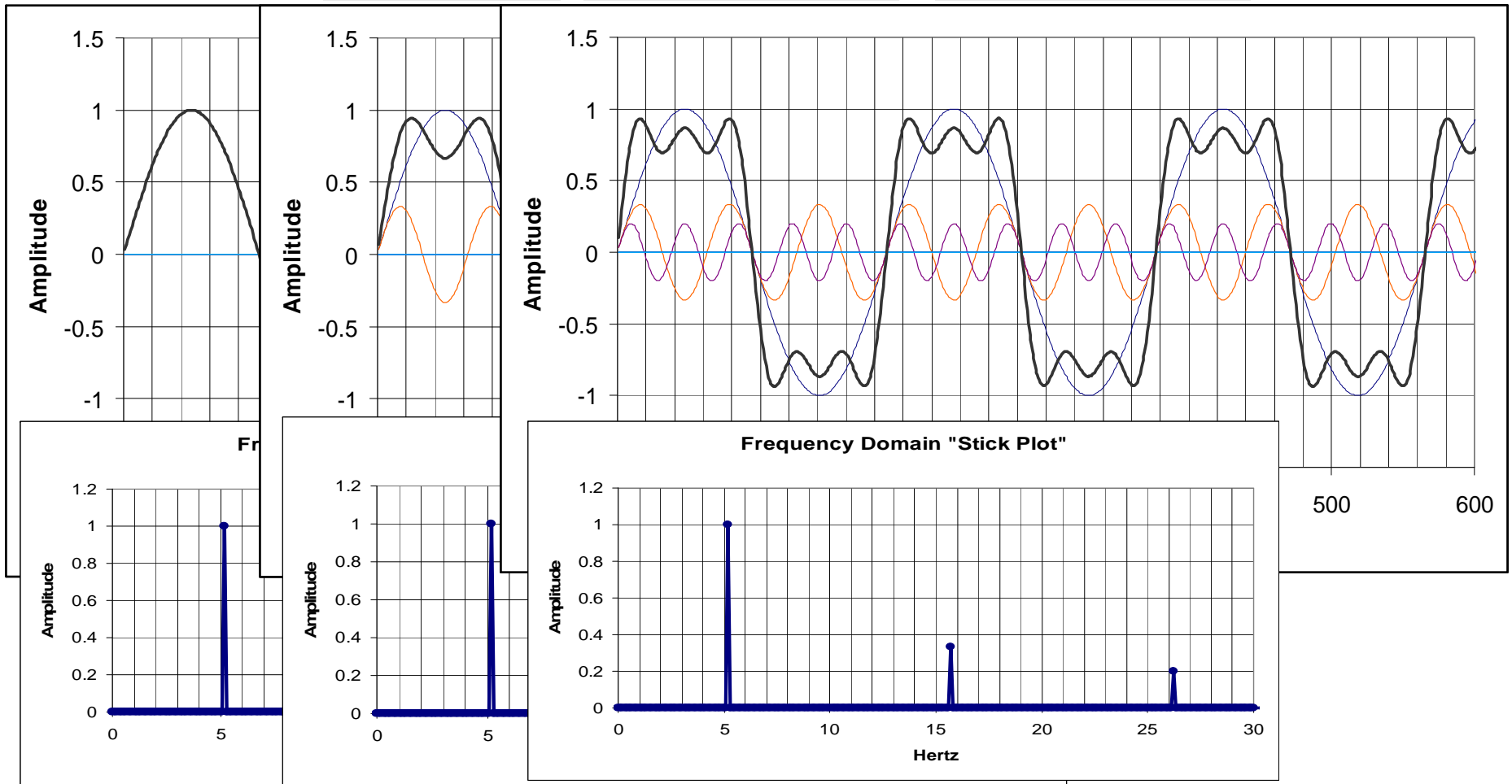
Fundamental

Third Harmonic

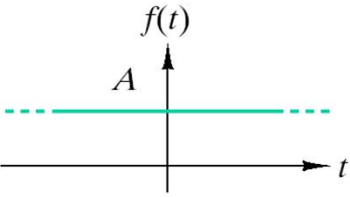
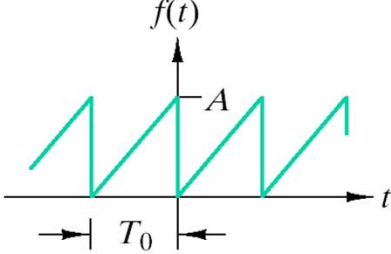
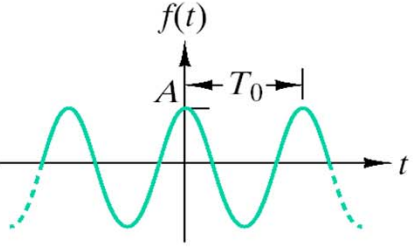
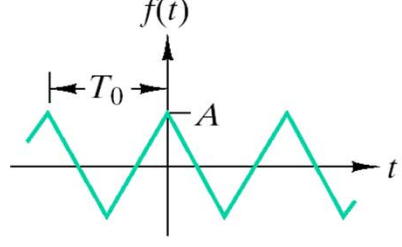
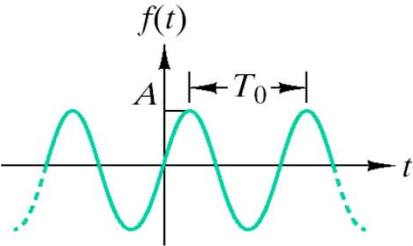
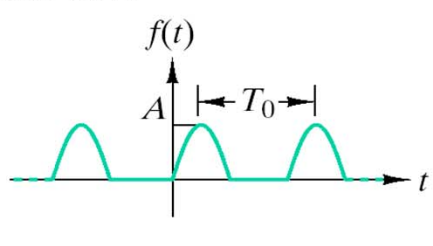
Fifth Harmonic

Constructing a Square Wave

$$V(t) = \frac{4A}{\pi} \left(\sin(\omega_0 t) + \frac{1}{3} \sin(3\omega_0 t) + \frac{1}{5} \sin(5\omega_0 t) + \dots \right)$$

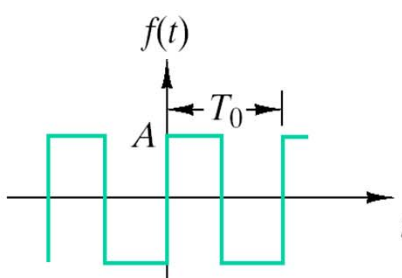
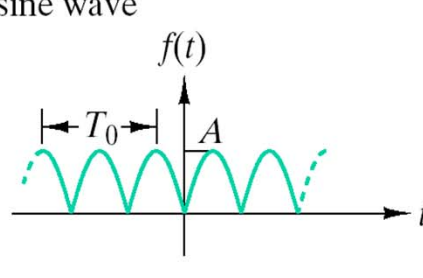
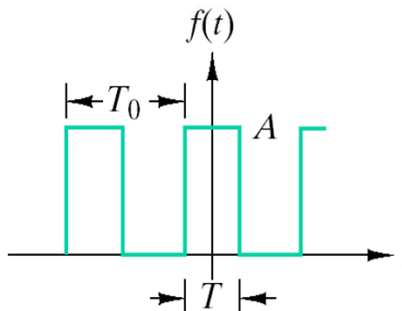
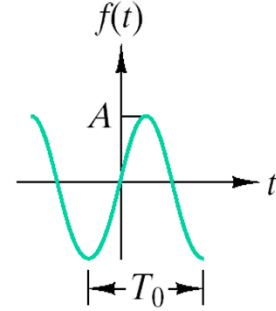


Some Fourier Coefficients¹

Waveform	Fourier Coefficients	Waveform	Fourier Coefficients
Constant (dc) 	$a_0 = A$ $a_n = 0$ all n $b_n = 0$ all n	Sawtooth wave 	$a_0 = \frac{A}{2}$ $a_n = 0$ all n $b_n = -\frac{A}{n\pi}$ all n
Cosine wave 	$a_0 = 0$ $a_1 = A$ $a_n = 0$ $n \neq 1$ $b_n = 0$ all n	Triangular wave 	$a_0 = 0$ $a_n = \frac{8A}{(n\pi)^2}$ n odd $a_n = 0$ n even $b_n = 0$ all n
Sine wave 	$a_0 = 0$ $a_n = 0$ all n $b_1 = A$ $b_n = 0$ $n \neq 1$	Half-wave rectified sine wave 	$a_0 = \frac{A}{\pi}$ $a_n = \frac{2A/\pi}{1-n^2}$ n even $a_n = 0$ n odd $b_1 = \frac{A}{2}$ $n = 1$ $b_n = 0$ $n \neq 1$

¹Thomas and Rosa (2004). "The Analysis and Design of Linear Circuits," 4th Ed., John Wiley and Sons, Inc

More Fourier Coefficients¹

Waveform	Fourier Coefficients	Waveform	Fourier Coefficients
<p>Square wave</p> 	$a_0 = 0$ $a_n = 0 \quad \text{all } n$ $b_n = \frac{4A}{n\pi} \quad n \text{ odd}$ $b_n = 0 \quad n \text{ even}$	<p>Full-wave rectified sine wave</p> 	$a_0 = 2A/\pi$ $a_n = \frac{4A/\pi}{1 - n^2} \quad n \text{ even}$ $a_n = 0 \quad n \text{ odd}$ $b_n = 0 \quad \text{all } n$
<p>Rectangular pulse</p> 	$a_0 = \frac{AT}{T_0}$ $a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi T}{T_0}\right)$ $b_n = 0 \quad \text{all } n$	<p>Parabolic wave</p> 	$a_0 = 0$ $a_n = 0 \quad \text{all } n$ $b_n = \frac{32A}{(n\pi)^3} \quad n \text{ odd}$ $b_n = 0 \quad n \text{ even}$

¹Thomas and Rosa (2004). "The Analysis and Design of Linear Circuits," 4th Ed., John Wiley and Sons, Inc

Computing Discrete Fourier Transforms

- If there are N sampled points per period in time domain
- ⇒ Requires N Fourier components to fully represent
 - ⇒ Components a_n and b_n count as one Fourier frequency component
 - ⇒ Components can be expressed as
 - ⇒ $A(\omega) = |A(\omega)|\exp(i\phi(\omega))$
 - ⇒ $A(\omega)$ is complex

Requires $N \times N$ complex multiplies to compute discrete Fourier series of N sample long time series.

Fast Fourier Transform (FFT)

- Use math tricks to minimize number of multiplies
 - ⇒ $N \log_2(N)$ multiplies to compute Fourier Series
 - ⇒ Your scopes do FFTs

Filtered signal

Assume $V(t)$ is filtered by filter $T(f)$ to produce $V_{out}(t)$

Fourier series:
$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$f_0 = \omega_0 / (2\pi)$$

If filter $T(f)$ is real

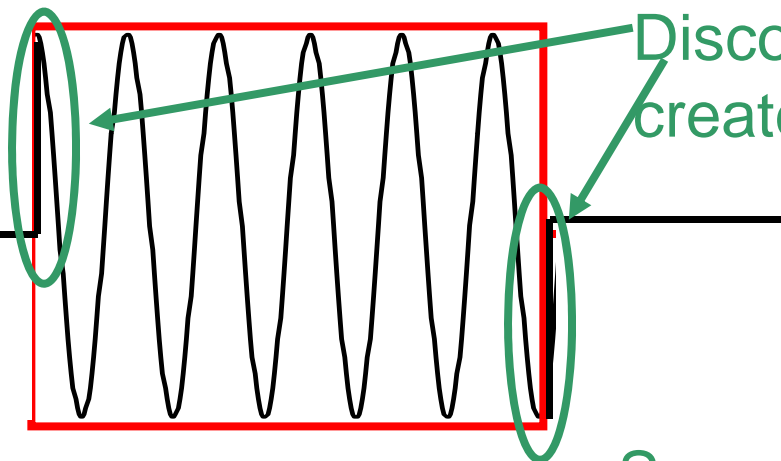
$$V_{out}(t) = a_0 T(0) + \sum_{n=1}^{\infty} T(nf_0) (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

If filter $T(f)$ is complex: $A(f)\exp(j\theta(f))$

$$V_{out}(t) = a_0 A(0) + \sum_{n=1}^{\infty} A(nf_0) \left(a_n \cos(n\omega_0 t + \theta(nf_0)) + b_n \sin(n\omega_0 t + \theta(nf_0)) \right)$$

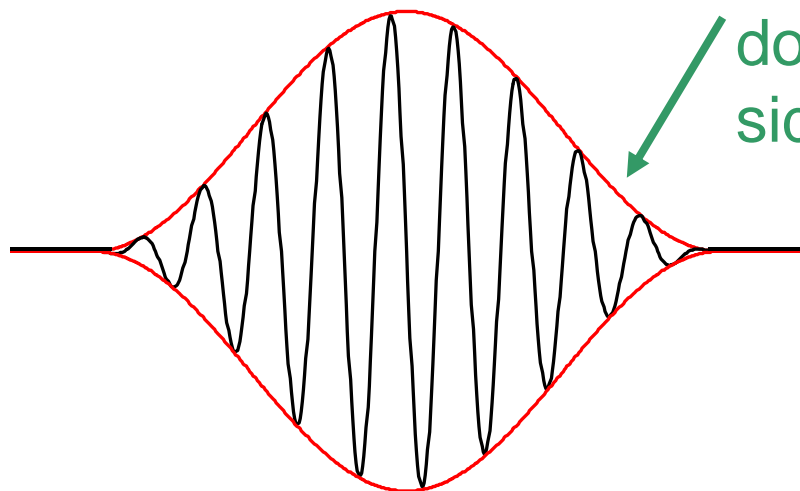
Windows for the FFT

Rectangular Window
(Boxcar)



Discontinuities
create sidebands

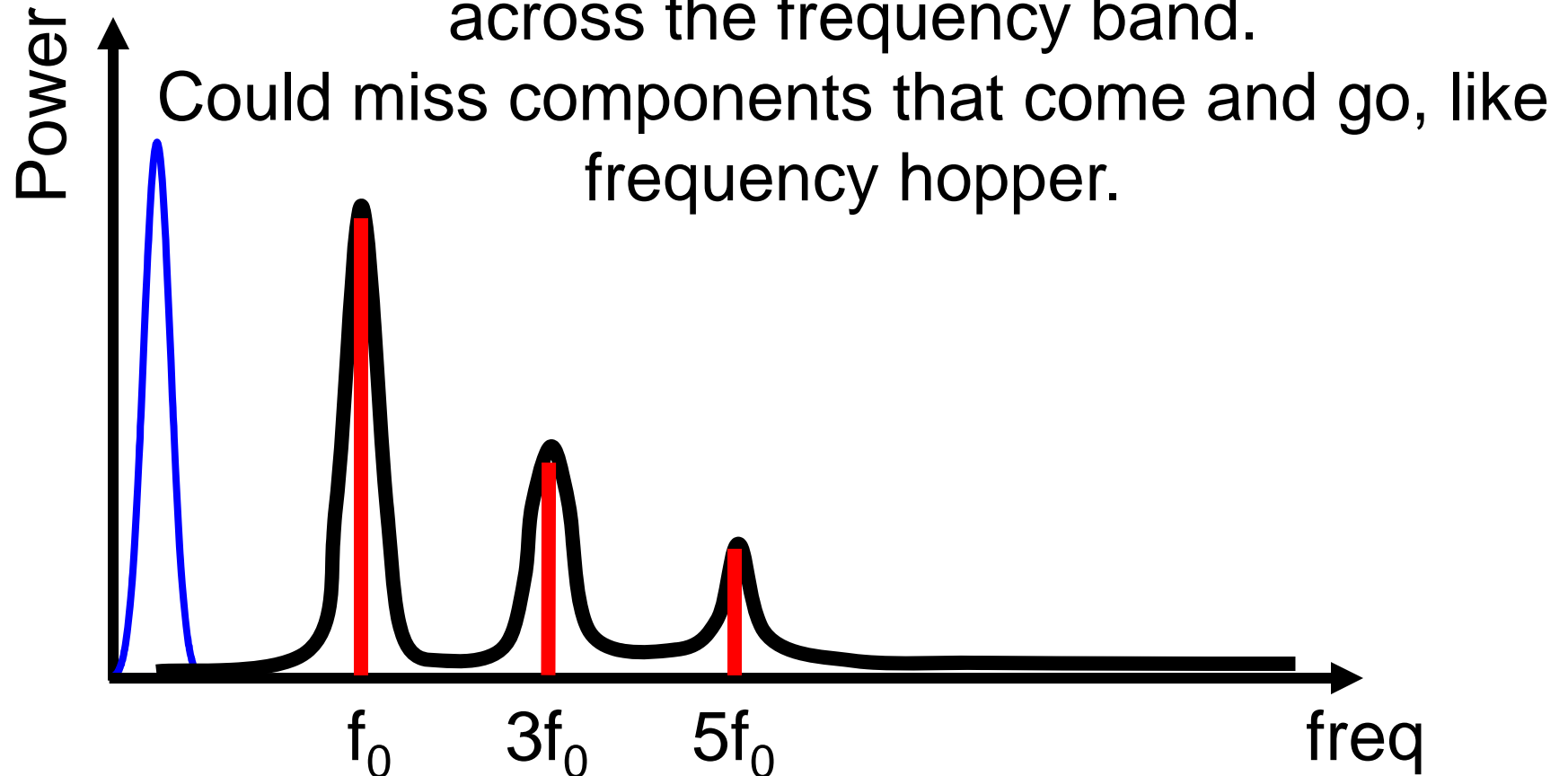
Hanning Window



Smooth up and
down limits
sidebands

Swept Spectrum Analyzer

Center frequency of a hi-Q filter is swept across the frequency band.

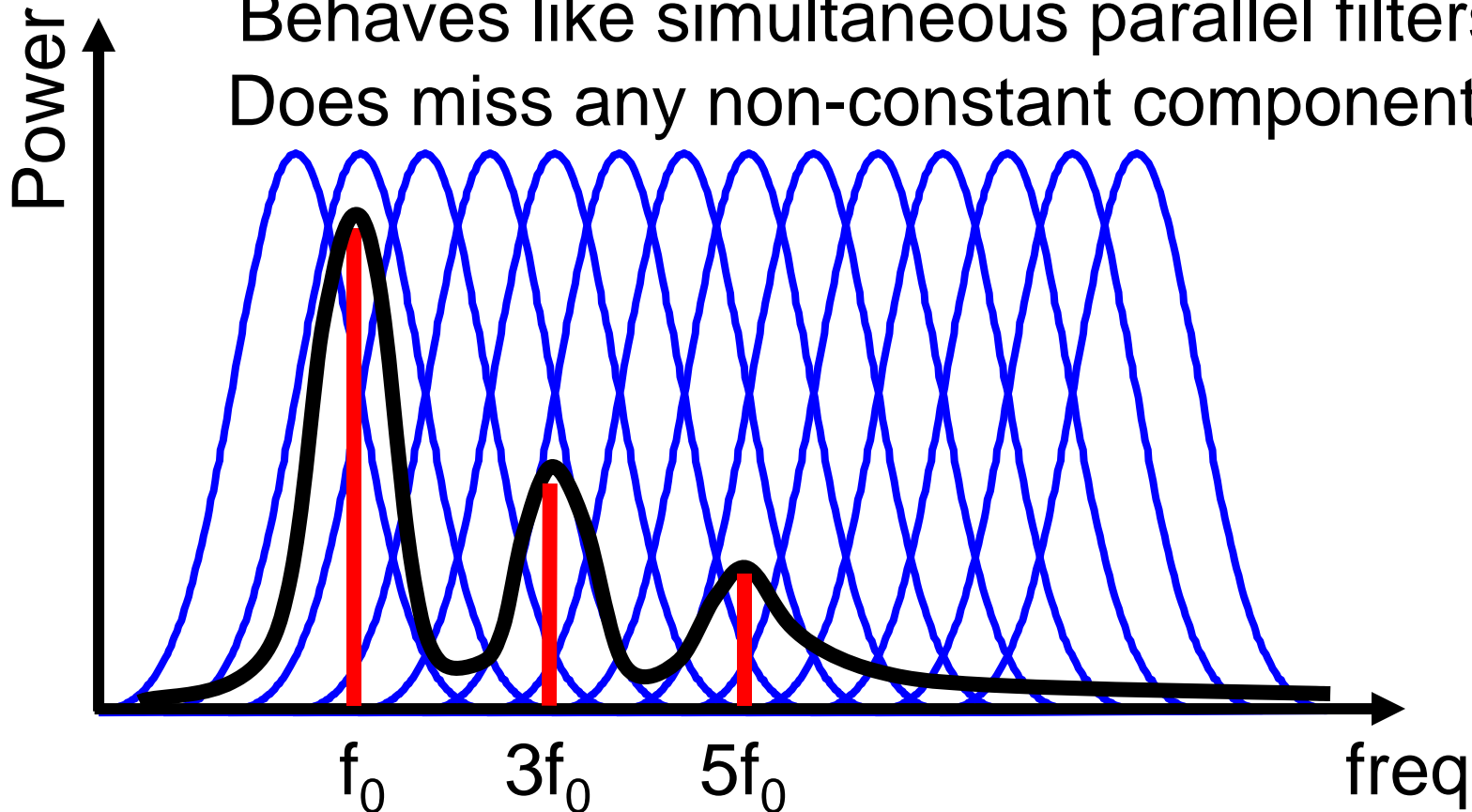


Good for high frequency signals. Typically expensive.
Depends on signals being repetitive.

Fast Fourier Transform Analyzer

Time domain signal is first digitized, then FFT is performed

Behaves like simultaneous parallel filters:
Does miss any non-constant components.

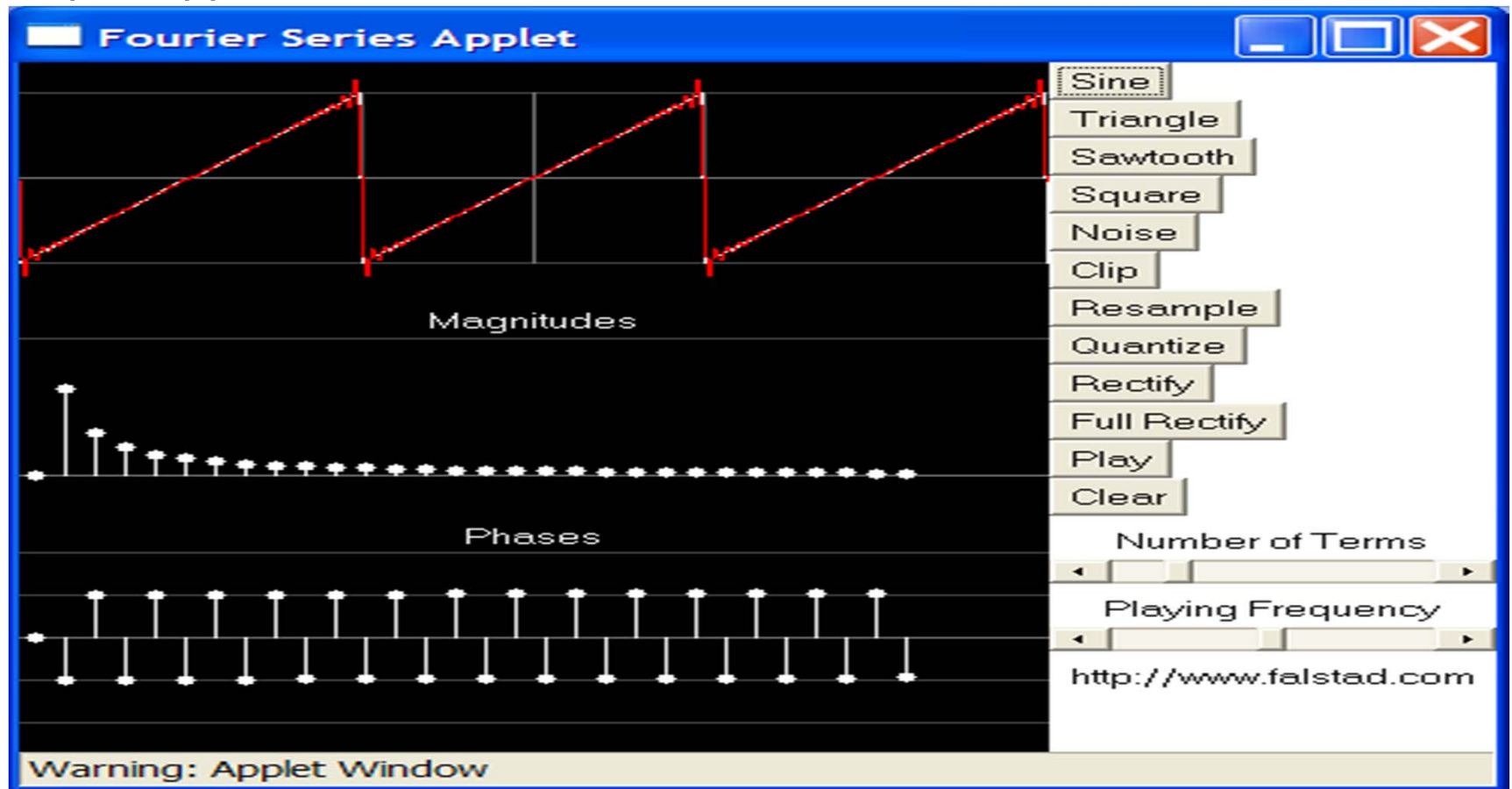


Captures full signal, but limited in bandwidth.

Low cost. Built into some oscilloscopes.

Fourier in the Audio

Helpful applet:



<http://www.falstad.com/fourier/>

References

1. Paul Horowitz and Winfield Hill (1989). “The Art of Electronics,” 2nd Ed., Cambridge, pages 1025-1038.
2. Roland E. Thomas and Albert J. Rosa (2004). “The Analysis and Design of Linear Circuits,” 4TH Ed., John Wiley and Sons
3. Paul Falstad, “some applets ... to help visualize various concepts in math and physics”, <http://www.falstad.com/mathphysics.html>, 15 Feb 2010
4. “Efunda, Engineering Fundamentals” web site; accessed 15 Feb 2010
http://www.efunda.com/designstandards/sensors/methods/DSP_nyquist.cfm
5. “The Fundamentals of FFT-Based Signal Analysis and Measurement in LabVIEW and LabWindows”; 15 Feb 2010, <http://zone.ni.com/devzone/cda/tut/p/id/4278>