1. An arbitrary EMF $\mathcal{E}(t)$ is applied to the LRC circuit shown above.

(a) Write an inhomogeneous ordinary differential equation for $V(t)$ in the LRC circuit shown above.

(b) Find the general solution $V_C(t)$ to the complementary equation (that is, for $\mathcal{E}(t) = 0$). In this and subsequent parts, assume that the system is overdamped. You may find it convenient to define

$$a = \frac{R}{2L}, \quad b = \frac{\sqrt{R^2C^2 - 4LC^2}}{2LC}.$$

(c) The circuit is initially quiescent, $V = 0$ for $t < 0$. Find the impulse response of the circuit: $V = G(t, t_0)$ for $\mathcal{E}(t) = \delta(t - t_0), t_0 > 0$. This is also called the Green’s function.

(d) Find the response to a square pulse,

$$\mathcal{E}(t) = \begin{cases} 
0, & t < 0; \\
\mathcal{E}_0, & 0 \leq t \leq T;
0, & t > T.
\end{cases}$$
2. A “trailer” of mass $m$, at position $x$, is attached to a “trailer hitch,” modeled by a spring. The motion $y(t)$ of the other end of the spring is determined by the driver of the “pickup truck,” which is towing the trailer. The coordinates are chosen so that the spring is relaxed when $x = y$. The initial conditions are

$$x(0) = 0, \quad \dot{x}(0) = v_0.$$ 

(a) Write the equation of motion for $x(t)$ in terms of the natural frequency $\omega = \sqrt{k/m}$.

(b) Green’s method does not work with inhomogeneous boundary conditions such as this. Briefly explain why. Now choose a new coordinate $q = x - h(t)$ such that $h(t)$ satisfies the boundary conditions on $x$ (see the end of §15.2.5 in Riley, Hobson & Bence). Write an equation of motion for $q$. What are the new boundary conditions?

(c) Find the general solution $q_C(t)$ to the complementary (homogeneous) equation.

(d) Find the Green’s function for the new equation of motion.

(e) Use the Green’s function to find $q(t)$ and hence $x(t)$ for $t > 0$ in the trivial case of $y(t) = 0$. Check your answer.

(f) The trailer has, unfortunately, run away. It is heading down the street at speed $v_0$, without the pickup truck. The owner jumps into the truck and pursues the trailer. At time $t = 0$, the trailer hitch is snagged by the passing pickup, whose trajectory is given by $y = v_1te^{-bt}$. Find $q(t)$ and $x(t)$ for $t > 0$. 
