APPLICATION: SOLUTION OF LINEAR ODE'S  

\[ \text{NO CURRENT HERE!} \]

\[ E(t) \quad \text{APPLIED EMF} \]
\[ V(t) \quad \text{OUTPUT VOLTAGE} \]

WHAT'S THE ODE FOR \( V(t) \) ?

\[ E - V = iR \quad \Rightarrow \quad i = CV \]

\[ E - V = RC \dot{V} \]

\[ RC \dot{V} + V = E(t) \]

NOW, FOURIER TRANSFORM BOTH SIDES:

\[ \tilde{V}(w) \left[ i w RC + 1 \right] = \tilde{E}(w) \]

\[ \tilde{V}(w) = \frac{1}{1 + i w RC} \tilde{E}(w) \]

IN FOURIER SPACE, THE ODE IS A SIMPLE, ALGEBRAIC EQUATION.

THE QUANTITY

\[ \Phi(w) = \frac{1}{1 + i w RC} = \frac{1 - i w RC}{1 + (wRC)^2} = A(w) e^{i \phi(w)} \]

IS THE FREQUENCY RESPONSE OF THE SYSTEM. IT HAS A MAGNITUDE (GAIN OR ATTENUATION) AND A PHASE:

\[ A = \frac{1}{\sqrt{1 + (wRC)^2}} \]
\[ \phi = -\tan^{-1} wRC \]
The solution for $V(t)$ is:

$$V(t) = \mathcal{F}^{-1} \left[ \frac{\tilde{E}(w)}{1 + i\omega RC} \right]$$

Now, if $E(t) = E_0 e^{i\omega_f t}$, things become very straightforward:

$$\tilde{E}(w) = \sqrt{2\pi} E_0 \delta(w - \omega_f)$$

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \frac{\sqrt{2\pi} E_0 \delta(w - \omega_f)}{1 + i\omega RC} \, dw$$

$$= \frac{E_0 e^{i\omega_f t}}{1 + i\omega RC} = \frac{E_0 \tilde{E}(w)}{R(w)}$$

That is, $V = E A \tilde{E}(w)$ if $E(t)$ is harmonic.

$\rightarrow$ In this case, the phase $\phi = -\tan^{-1} RC < 0$,
which means that $V(t)$ lags $E(t)$.

Phase lag $(-\phi)$
The filter suppresses high frequencies, with response:

\[ A = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad \text{(Lorentzian)} \]

\( A(\omega) \)

\( \frac{1}{\omega RC} \) describes \( \omega \gg \frac{1}{RC} \) behavior.

You'll solve an ODE in Exercise 13.5.
In HW and Class Examples, we have been ignoring the homogeneous soln. of the ODE.

\[ m \ddot{x} = -k x + f(t) \]

\[ \tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{k - m \omega^2} \]

This seems to imply that if \( f = 0 \), \( x = 0 \).

Consider:

\[ m \ddot{x} + k x = f(t) = 0 \]

\[ (k - m \omega^2) \tilde{x}(\omega) = 0 \]

\[ \tilde{x} = 0 \text{ or } k = m \omega^2 \]

\[ \omega = \pm \sqrt{\frac{k}{m}} = \omega_0 \]

i.e., \( x = A e^{i \omega_0 t} + B e^{-i \omega_0 t} \)

Since the equation is linear,

\[ x(t) = \mathcal{F}^{-1} \left( \frac{\tilde{f}(\omega)}{k - m \omega^2} \right) + A e^{i \omega_0 t} + B e^{-i \omega_0 t} \]

Adding the homogeneous solution allows us to have any initial condition, independent of forcing.
CONSEQUENCE: THE FULL SOLN. MUST INCLUDE THE HOMOGENEOUS PIECE:

\[ \tilde{x}(w) = \frac{\tilde{f}(w)}{k - mw^2} + A \delta(w - \sqrt{\frac{k}{m}}) + B \delta(w + \sqrt{\frac{k}{m}}) \]

Q: WHAT IF \( f(t) = \cos \sqrt{\frac{k}{m}} t \) ?

Q: WHAT DOES THIS IMPLY ABOUT THE SOLUTION TO OUR RC CIRCUIT?