FOURIER TRANSFORMS SO FAR

- Definition
- Normalization (and connection with AM)
- Inverse
- Properties
  - Linearity
  - Differentiation
  - Integration
  - Scaling
  - Translation
  - Exponential Multiplication
- Examples: $f(x) \rightarrow \text{something}$
- Symmetries - Odd & Even Functions
- Solution of Linear ODE's (2 Examples)

Today
- Finish Friday's last example
- HW?

Convolution
- Conv Theorem
  - Example: How to build a Gaussian from anything
- Cross-correlation
CONVOLUTION §13.1.7

\[ f(t) \ast g(t) = \int_{-\infty}^{\infty} f(x) g(t-x) \, dx \]

\[ g \ast f = \int_{-\infty}^{\infty} g(x) f(t-x) \, dx \]

Let \( z = t - x \)

\[ dz = -dx \]

\[ g \ast f = -\int_{z=\infty}^{-\infty} g(t-z) f(z) \, dz \]

\[ = \int_{-\infty}^{\infty} f(z) g(t-z) \, dz \]

\[ \therefore g \ast f = f \ast g \]

NOTE THAT \textbf{CONVOLUTION IS EXACTLY WHAT WE DO WITH GREEN'S FUNCTIONS:}

\textbf{WILL}

\textbf{GIVEN}

\[ \mathcal{D} u(x) = f(x) \]

\[ \mathcal{D} g(x-x_0) = \delta(x-x_0) \]

\textbf{RECALL}

\[ u(x) = \int_{\Omega} G(x-x_0) f(x_0) \, dx_0 = f \ast g. \]
Example

An amplifier has a pulse response

\[ P(t) = \begin{cases} \frac{A \tau}{t} e^{-t/\tau}, & t > 0 \\ 0, & t < 0 \end{cases} \]

So the response to an input signal \( S(t) \) is:

\[ R(t) = \int_{-\infty}^{\infty} S(t') P(t - t') \, dt' \]

a) Find the gain at DC: \( g_0 \)

If \( S(t) = 1 \), \( R(t) = g_0 \)

\[ g_0 = \int_{-\infty}^{\infty} P(t - t') \, dt' = \int_{0}^{\infty} A t e^{-t/\tau} \, dt \]

\[ u = t \quad dv = e^{-t/\tau} \, dt \quad \int uv = uv - \int vu \]

\[ du = dt \quad v = -\tau e^{-t/\tau} \]

\[ g_0 = -A \tau e^{-t/\tau} \bigg|_{0}^{\infty} + A \tau e^{-t/\tau} \int_{0}^{\infty} \, dt \]

\[ = 0 - A \tau e^{-t/\tau} \bigg|_{0}^{\infty} = A \tau^2 \]

\[ g_0 = A \tau^2 \]

b) What is the minimum time between resolved pulses?
Now, what happens if we Fourier transform the convolution?

\[ f \ast g = \int_{-\infty}^{\infty} f(t') g(t-t') dt' \]

\[ \mathcal{F}(f \ast g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' e^{-i\omega t} f(t') g(t-t') \]

\[ = \int_{-\infty}^{\infty} dt' f(t') \mathcal{F}[g(t-t')] \]

\[ = \int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'} \hat{g}(\omega) \]

\[ \mathcal{F}(f \ast g) = \sqrt{2\pi} \hat{f}(\omega) \hat{g}(\omega) \quad (13.38) \]

**Convolution Theorem.**

So convolution in real space is multiplication in frequency space!

Similarly, \( \frac{1}{\sqrt{2\pi}} \hat{f}(\omega) \ast \hat{g}(\omega) = \mathcal{F}(f(t)g(t)) \quad (13.38) \)

1.6. \[ \mathcal{F}^{-1}(\hat{f} \ast \hat{g}) = \sqrt{2\pi} fg \]
IMPLICATION FOR GREEN'S FUNCTIONS:

Suppose again that

\[ D u(t) = f(t) \]

and we have a Green's function \( G(t-t_0) \), so

\[ u(t) = \int_{-\infty}^{\infty} G(t-t_0) f(t_0) dt_0 = f \ast G. \]

Then \( \tilde{u}(\omega) = \tilde{f}(\omega) \tilde{g}(\omega) \)

and \( \tilde{g}(\omega) = \frac{\tilde{u}}{\tilde{f}} \) \( \text{system response} \)

and \( \tilde{f}(\omega) \) \( \text{forcing} \)

In other words, \( \tilde{g}(\omega) \) is the frequency response!

So Green's functions and the Fourier transform method for solving differential equations are actually highly related:

→ G.F. asks response to \( s(t-t_0) \)

→ F.T. asks response to \( s(\omega-w_0) \)
EXAMPLE: USEFULNESS OF CONVOLUTION IN PROBABILITY THEORY

Suppose we have two statistically independent random variables,

\[ \alpha_1, \alpha_2. \]

with PDF's \( P_1(\alpha_1), P_2(\alpha_2). \)

Let \( x = \alpha_1 + \alpha_2. \) What is \( P(x) \)?

The probability of getting \( x \) for a specific \( \alpha_1 \) is:

\[ P(x \mid \alpha_1) = P_2(\alpha_2), \quad \text{where } x = \alpha_1 + \alpha_2, \text{ i.e.} \]

\[ = P_2(x - \alpha_1). \]

Of course, \( \alpha_1 \) can be anything, and is distributed as given by \( P_1 \). So the probability of getting \( x \) regardless of the value of \( \alpha_1 \) is:

\[ P(x) = \int_{-\infty}^{\infty} P(x \mid \alpha_1) P_1(\alpha_1) \, d\alpha_1, \]

\[ = \int_{-\infty}^{\infty} P_1(\alpha_1) P_2(x - \alpha_1) \, d\alpha_1, \]

\[ P(x) = P_1(x) \times P_2(x) \]
Example: Central Limit Theorem (RHB § 36.10)

Theorem

Suppose $x_i^i, i = 1, 2, \ldots, N$, are independent random variables, with PDF's $p_i(x_i)$, with mean $\mu_i$ and variance $\sigma_i^2$.

The random variable $X = \sum_i x_i$ has the properties:

i. $\langle X \rangle = \sum_i \mu_i$

ii. Variance $\langle [X - \langle X \rangle]^2 \rangle = \sum_i \sigma_i^2$

iii. As $N \to \infty$, $P(X)$ tends to a Gaussian with the NO mean & variance given above.

In what follows, I want the $N \to \infty$ behavior as "nice" as possible. Let's specialize to a simpler case:

- Let $f_i(x_i) = f(x_i)$ (all $N$ variables with the same distro)

- Choose $f(x_i)$ so that $\mu = 0$ ($\langle x_i \rangle = 0$)

- Choose $f(x_i)$ so that $\sigma^2 = \frac{1}{N}$, and thus

  $\langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle = \sum_{i=1}^{N} \frac{1}{N} = 1$

We will then expect to find

$$\lim_{N \to \infty} P(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

convolutions
OR, using the convolution theorem, \( \lim_{N \to \infty} \left[ \hat{f}(k) \right]^N = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \).

This is a magical property, if true!

- Pick an arbitrary \( f(x) > 0, \int f(x) dx = 1 \)
- Tweak so that \( \mu = 0 \) and \( \sigma = 1/\sqrt{N} \)

Then \( \hat{f}(k) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \)

Q: What are the requirements on \( f \), so that this works?

Q: What are the requirements on \( f \) so that

\[ \hat{f}^N(k) \rightarrow A e^{-\frac{(k-k_0)^2}{2\sigma^2}} \]

for some \( A, k_0, \sigma \)?
For fun, let's try out

\[ p_1(\alpha) = \frac{1}{2\alpha} \ e^{-|\alpha|/\alpha} \]

(\(\alpha\): is the normalization correct?)

\[ \tilde{p}_1(\delta) = \frac{1}{1 + \alpha^2 \delta^2} \]

Convolving \(N\) copies of \(P(\alpha)\) leads to \(P_N(\alpha)\):

\[ P_N(\delta) = \tilde{p}_1^N = \frac{1}{\left(1 + \alpha^2 \delta^2\right)^N} \]

\[ = \frac{1}{1 + N\alpha^2 \delta^2 + \frac{N(N-1)}{2} \alpha^4 \delta^4 + \frac{N(N-1)(N-2)}{6} \alpha^6 \delta^6 + \ldots} \]

To keep \(\sigma\) finite as \(N \to \infty\), let \(\alpha = \frac{1}{\sqrt{N}}\).

Now,

\[ \tilde{p}_N = \frac{1}{1 + \delta^2 + \frac{\delta^4}{2} + \frac{\delta^6}{6} + \frac{\delta^8}{24} + \ldots} \]

\[ = \frac{1}{e^{\delta^2}} = e^{-\delta^2} \]

Q: Suppose I choose \(P(\alpha) = \frac{A}{1 + \alpha^2 \alpha^2}\),

so \(\tilde{p}_1(\delta) \propto e^{-\delta^2/\alpha}\).

Is \(\tilde{p}_1^N\) a Gaussian? Explain!