LAST TIME, I SHOWED YOU HOW TO SOLVE A LINEAR, INHOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS USING THE LAPLACE TRANSFORM:

DIFFERENTIAL EQUATION:

\[ D x(t) = f(t) \]

FORCED

UNKNOWN

TAKE THE TRANSFORM,

\[ L(Dx) = F(s) \]

AND WE OBTAIN AN ALGEBRAIC EQUATION:

\[ G(s) \bar{x}(s) = \bar{f}(s) \]

\[ \bar{x}(s) = \frac{F(s)}{G(s)} = \bar{G}(s) \bar{f}(s) \]

WHERE

\[ \bar{G}(s) = \frac{1}{g(s)} = L(D^{-1}x(t)) \]

THE USUAL SOLUTION (SEE PROB. 2) IS TO TAKE \( L^{-1} \) OF BOTH SIDES. IF INSTEAD WE USE THE CONVOLUTION THEOREM, GREEN'S FUNCTION.

\[ x(t) = G(t) * f(t) \]

AND THE SAME ARGUMENT CAN BE USED WITH THE FOURIER TRANSFORM.

WHAT DID I LEAVE OUT OF THE MATH JUST NOW?

- INITIAL CONDITIONS FROM \( L(x) \)
- THE CONVOLUTION THM FOR \( L \) IS NOT EXACTLY CONVOLUTION.
How would the above differ for a Fourier transform approach?

This has been a kind of lead-in for Green's functions, RH8 §15.2.5.

In what follows, we set aside the integral transforms and derive the Green's function approach from entirely different considerations.
**INTRODUCTION TO GREEN'S FUNCTIONS — CONCEPTUAL...**

Suppose we have an inhomogeneous, linear ODE,

\[ D_y(x) = f(x), \quad x \in \Omega \]  \hspace{1cm} (1)

Suppose further that we have a general solution to the complementary (homogeneous) equation,

\[ D_y^c = 0. \]

It is then only a little harder to solve

\[ D_y G(x, z) = S(x - z), \quad z \in \Omega. \]  \hspace{1cm} (2)

- Same as complementary equation when \( x < z \) or \( x > z \).
- Requires careful matching across \( x = z \) boundary.

Then if we integrate (2), times our forcing \( f(z) \),

\[ \int_\Omega d\xi \ f(\xi) D_y^c G(x, \xi) = \int_\Omega d\xi \ f(\xi) S(x - \xi) \]

\[ \Rightarrow D_y \int_\Omega d\xi \ f(\xi) G(x, \xi) = f(x). \hspace{1cm} \text{(cf. (1))} \]

Thus, the solution to (1) is

\[ y(x) = \int_\Omega d\xi \ f(\xi) G(x, \xi). \]

Now let's do an example with homogeneous boundary conditions...
EXAMPLE

\[ Y'' + \frac{1}{4} Y = f(x), \quad (3) \]

B.C.'s: \[ Y(0) = Y(\pi) = 0 \]

HOMOGENEOUS SOLN:

\[ Y'' + \frac{1}{4} Y = 0 \]

\[ Y = A \sin \frac{x}{2} + B \cos \frac{x}{2} \]

GREEN'S FUNCTION:

WRITE \[ f(x) = \int_0^\pi f(z) S(x - z) \, dz \]

\[ y(x) = \int_0^\pi f(z) G(x, z) \, dz \]

DOES \( y(x) \) SATISFY THE B.C.'S? \( (3) \) BECOMES:

\[ \int_0^\pi f(z) \left[ \frac{\partial}{\partial x} G + \frac{1}{4} G \right] \, dz = \int_0^\pi f(z) S(x - z) \, dz \]

OR: \[ \frac{\partial}{\partial x} G + \frac{1}{4} G = S(x - z) \quad (4) \]

REGION 1: \( x < z \)

\[ G_1(x, z) = A \sin \frac{x}{2} + B \cos \frac{x}{2}, \quad x < z \]

\[ G_1(0, z) = 0 \Rightarrow B = 0. \]

\[ G_1(x, z) = A \sin \frac{x}{2} \]

REGION 2: \( x \geq z \)

\[ G_2(x, z) = C \sin \frac{x}{2} + D \cos \frac{x}{2} \]

\[ G_2(\pi, z) = 0 \Rightarrow C = 0 \]

\[ G_2(x, z) = D \cos \frac{x}{2} \]

NOW, WHAT SHOULD HAPPEN AT \( x = z \)?
Now, we want \( G(x, z) = \begin{cases} G_1, & x \leq z \\ G_2, & x > z \end{cases} \) to satisfy Equation (4) in the neighborhood of \( x = z \).

It is clear that one of the LHS terms must contain \( s(x-z) \). It must be the highest derivative, if it were not the highest derivative term, then the derivatives higher than the one containing \( s(x-z) \) would contain derivatives of \( s(x-z) \), and it would be impossible to satisfy the ODE.

Since there is a \( s(x-z) \) in \( \frac{\partial G}{\partial x} \), there must also be a discontinuity in the next lower derivative, \( \frac{\partial G}{\partial x} \). Lower derivatives are continuous!

Let's integrate (4) over an infinitesimal neighborhood centered on \( x = z \):

\[
\int_{z-\delta}^{z+\delta} dx \left[ \frac{\partial G}{\partial x} + \frac{1}{4} G \right] = \int_{z-\delta}^{z+\delta} dx \ s(x-z), \quad \delta \to 0.
\]

This term contains \( s(x-z) \) and does not contribute.

\[
\frac{\partial G}{\partial x} \bigg|_{x = z-\delta}^{x = z+\delta} = 1, \quad \text{or --}
\]

\[
\frac{\partial G_2}{\partial x} \bigg|_{x = z} - \frac{\partial G_1}{\partial x} \bigg|_{x = z} = 1.
\]

\[-\frac{1}{2} D \sin \frac{z}{2} - \frac{1}{2} A \cos \frac{z}{2} = 1 \quad (5)\]

We insist upon continuity for lower order terms. In this case, only \( G \) remains:

\[
G_1 (z, z) = G_2 (z, z)
\]

\[
\therefore \quad A \sin \frac{z}{2} = D \cos \frac{z}{2} \quad (6)
\]

\[KE = A\]
Combining (5) & (6), we solve for A & D:

\[-\frac{1}{2} D \sin \frac{x}{2} - \frac{1}{2} D \cot \frac{x}{2} \cos \frac{x}{2} = 1\]

\[-\frac{1}{2} D \left\{ \sin \frac{x}{2} + \cos \frac{x}{2} \right\} = \sin \frac{x}{2} \]

\[D = -2 \sin \frac{x}{2}\]

\[A = -2 \cos \frac{x}{2}\]

The Green's function is thus:

\[G(x, \xi) = \begin{cases} -2 \cos \frac{x}{2} \sin \frac{\xi}{2}, & x < \xi \\ -2 \sin \frac{x}{2} \cos \frac{\xi}{2}, & x \geq \xi \end{cases}\]

And the solution to the inhomogeneous ODE (3) is:

\[y(x) = \int_{0}^{\pi} f(\xi) G(x, \xi) \, d\xi \]

\[= -2 \cos \frac{x}{2} \int_{0}^{x} f(\xi) \sin \frac{\xi}{2} \, d\xi - 2 \sin \frac{x}{2} \int_{x}^{\pi} f(\xi) \cos \frac{\xi}{2} \, d\xi.\]

\(x \geq \xi, \text{ use } G_2\)

\(x < \xi, \text{ use } G_1\)