WE ARE ON CH. 20.

LAST TIME:

DISCUSSED VARIOUS PHYSICS PDES AND WHERE THEY COME FROM.

THIS TIME:

§§ 20.2-20.3, 1, FIRST-ORDER EQUATIONS
(A PRELUDE TO THE METHOD OF CHARACTERISTICS)

NEXT TIME:

§ 20.3.2 INHOMOGENEOUS FIRST-ORDER
§ 20.3.3 SECOND-ORDER EQUATIONS
§ 20.2 (My presentation is similar to RHB, but they give a different example)

Consider the first-order ODE

\[ x \frac{du}{dx} + y \frac{du}{dy} = 0. \]

I assert that the following are particular solutions:

\[ U_1 = \frac{x^2 + y^2}{xy} \]

\[ U_2 = \ln y - \ln x \]

\[ U_3 = \theta, \text{ where } x = r \cos \theta, \ y = r \sin \theta \]

\[ U_4 = \frac{z}{|z|}, \text{ where } z = x + iy. \]

→ Divide into 3-4 groups to check these solutions.

Question: What do these four solutions have in common?

Answer: All are of the form

\[ U = f(p), \text{ where } p(x,y) = y/x. \]

This is, in fact, the general solution for §.!
HOW DID I CONTRIVE THIS EXAMPLE? RHBill S RECIPE:

- CHOOSE \( p(x,y) = \frac{y}{x} \), ANY FUNCTION WILL DO! (BOOK USES \( x^2 + 2y \) AS AN EXAMPLE.)

- LETTING \( U = f(p) \) FOR SOME ARBITRARY \( f \),

\[
\begin{align*}
\frac{\partial u}{\partial x} &= f' \frac{\partial p}{\partial x} = -\frac{y}{x^2} f' \\
\frac{\partial u}{\partial y} &= f' \frac{\partial p}{\partial y} = \frac{f'}{x}
\end{align*}
\]

- WE WANT AN ODE THAT WORKS FOR ANY \( f(p) \).

TO ELIMINATE \( f' \) FROM THE ABOVE, DIVIDE THE TWO EQUATIONS (OR CROSS-MULTIPLY THEM):

\[
\frac{\partial u}{\partial x} \cdot \frac{\partial p}{\partial y} = \frac{\partial u}{\partial y} \cdot \frac{f'}{\partial x}
\]

WHICH, IN OUR CASE, IS

\[
\frac{1}{x} \frac{\partial u}{\partial x} + \frac{y}{x^2} \frac{\partial u}{\partial y} = 0
\]

\[
\frac{1}{x} \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0
\]

- IT ONLY REMAINS TO CHOOSE SOME PARTICULAR SOLUTIONS FOR THE EXAMPLE:

\[
U_1 = p + \frac{1}{p} = \frac{y}{x} + \frac{x}{y} = \frac{x^2 + y^2}{xy}
\]

\[
U_2 = ln \left( \frac{y}{x} \right) = ln y - ln x
\]

\[
U_3 = tan^{-1} \left( \frac{y}{x} \right) = \Theta \quad (IN \ \text{POLAR COORD'S})
\]

\[
U_4 = e^{i\Theta} = \cos \Theta + i\sin \Theta = \frac{x + iy}{\sqrt{x^2 + y^2}} = \frac{z}{|z|}
\]
§20.3.1 Solving First-Order Eq's (2 Independent Var's)

General Form:

\[ A(x,y) \frac{du}{dx} + B(x,y) \frac{du}{dy} + C(x,y) u = R(x,y) \]  \hspace{1cm} (20.9)

Note: If either of these is zero, it's an ODE. But the "constant" of integration is an arbitrary function of \( x \) (if \( A = 0 \)) or of \( y \) (if \( B = 0 \)). (See example in text)

First, consider if \( C = R = 0 \). If \( u = f(p) \), then (as in our example, Eq. \( \star \star \)),

\[ \frac{du}{dx} = f' \frac{dp}{dx} \]

\[ \frac{du}{dy} = f' \frac{dp}{dy} \]

Sub into (20.9):

\[ A f' \frac{dp}{dx} + B f' \frac{dp}{dy} = 0 \]

\[ A \frac{dp}{dx} + B \frac{dp}{dy} = 0, \quad p = p(x,y) \]  \hspace{1cm} (20.10)

Got our original PDE back. What good is that?!
SEEMINGLY INNOCENT QUESTION:
WHAT DOES IT TAKE TO HOLD \( p \) (AND THUS \( f \)) CONSTANT WHILE \( x \) AND \( y \) CHANGE?

\[
dp = \frac{dp}{dx} dx + \frac{dp}{dy} dy = 0 \quad (20.11)
\]

COMBINING 20.10 WITH 20.11 (DIVIDE EQ'S),

\[
\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}. \quad (20.12)
\]

IF WE INTEGRATE THIS EQ., THEN WE WILL GET AN INTEGRATION CONSTANT \( C \):

\[
BLAH(x,y) = C
\]

ANSWER: IF \( BLAH(x,y) \) IS CONSTANT, THEN \( p \) WILL BE CONSTANT. WE THEREFORE MAKE THE IDENTIFICATION

\[
p = BLAH(x,y)
\]

OR, IF MORE CONVENIENT,

\[
p = FUNCTION(BLAH(x,y)).
\]

DISCUSS: THE ARGUMENT ON THIS PAGE MAY SEEM MORE COMPELLING IF IT IS REVIEWED IN REVERSE ORDER.

NOTE: THIS PROCEDURE IS NOT GUARANTEED TO WORK. NOT ALL LINEAR PDE'S HAVE SOLN'S OF FORM \( f(p) \).
EXAMPLE (BOOK HAS A DIFFERENT ONE)

\[ x \frac{du}{dx} + y \frac{du}{dy} = 0 \quad \text{(SAME PDE AS BEFORE)} \]

FIRST, LET'S DO IT WITHOUT B.C.'S, TO GET A GENERAL SOLN.

APPLY EQ. 20.12:

\[
\int \frac{dx}{x} = \int \frac{dy}{y}
\]

\[ C + \ln x = \ln y \]

\[ C = \ln y - \ln x = \ln \left( \frac{y}{x} \right) \]

WE COULD LET \( p = \ln \left( \frac{y}{x} \right) \), BUT FOR SIMPLICITY, LET

\[ p = \frac{y}{x} \]

GEN SOLN: \( u = f(p) = f \left( \frac{y}{x} \right) \)

QUESTIONS?

\[ \longrightarrow \text{NOW APPLY B.C. : } u = 4y + 1 \text{ ON THE LINE } x = 1. \]
**Example**  *(you try!)*

Find General Solution of

\[
\frac{\partial u}{\partial x} = x \frac{\partial u}{\partial y}
\]

*(Answer: \( p(x,y) = x^2 + 2y \))*

**Boundary Conditions:**

(i) \( u = y \) on \( x = 1 \)

(ii) \( u = y \) on \( y = -x^2 \) *(Slightly trickier!)*