LAST TIME, WE CONSIDERED EQUATIONS OF THE FORM

\[ A(x,y) \frac{du}{dx} + B(x,y) \frac{du}{dy} = 0. \]

WE ASSUMED A SOLUTION OF THE FORM

\[ u(x,y) = f(p), \quad \text{for some } p(x,y). \]

\( \text{THIS PRODUCED A PDE FOR } p, \text{ WITHOUT } f \text{ IN IT. (KEY POINT)} \)

WE FOUND \( p(x,y) \) BY SETTING \( dp = 0 \) AND FINDING

\[ \frac{dx}{A} = \frac{dy}{B} \]

\( \uparrow \text{ INTEGRATE} \)

\[ \text{BLAH}(x,y) = \text{CONST.} = p \]

OR \( p = F(\text{BLAH}) \) FOR CONVENIENCE,

TODAY, WE CONSIDER MORE GENERAL PROBLEMS:

\[ A(x,y) \frac{du}{dx} + B(x,y) \frac{du}{dy} + C(x,y) u = R(x,y) \]

FIRST \quad \text{SECOND}
ADD A TERM $C(x, y)u$:

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + Cu = 0$$

TRY OUR OLD TRICK:

$$U(x, y) = f(p(x, y))$$

$$\frac{\partial u}{\partial x} = f' \frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial y} = f' \frac{\partial p}{\partial y}$$

SO OUR PDE (*A*) BECOMES:

$$A f' \frac{\partial p}{\partial x} + B f' \frac{\partial p}{\partial y} + C f(p) = 0$$

**UH-OH!** WE CAN'T GET RID OF $f$. THIS IS A NO-GO.

NEW TRICK: $U(x, y) = h(x, y) f(p)$

NOW,

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} f(p) + h f' \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} f(p) + h f' \frac{\partial p}{\partial y}$$

SUBSTITUTING INTO (*A*), OUR PDE BECOMES

$$\left( A \frac{\partial h}{\partial x} + B \frac{\partial h}{\partial y} + C h \right) f(p) + \left( A \frac{\partial p}{\partial x} + B \frac{\partial p}{\partial y} \right) h f' = 0$$

IF $h(x, y)$ IS ANY NONZERO SOLN. OF (*A*), THIS TERM DISAPPEARS.

WE ARE LEFT WITH A C-LESS PDE:

$$A \frac{\partial p}{\partial x} + B \frac{\partial p}{\partial y} = 0$$
... which we know how to solve:

\[
\frac{dx}{A} = \frac{dy}{B} \quad \downarrow \text{Integrate}
\]

BLAH(x, y) = \text{const.}

\[ p = \text{BLAH or } p = F(\text{BLAH}). \]

**Example** (Book, p. 648) Find general soln:

\[ x \frac{du}{dx} + 2 \frac{du}{dy} - 2u = 0 \quad (20.15) \]

**Summary of soln:** \[ p = xe^{-y/2} \]

\[ U(x, y) = h(x, y) f(xe^{-y/2}) \]

For \( h(x, y) \) can choose, e.g., \( e^y \) or \( x^2 \). So

\[ U(x, y) = e^y f(xe^{-y/2}) \quad \{ \text{general solution} \} \]

or \( = x^2 g(xe^{-y/2}) \)

**Note that** \( g(p) = f(p)/p^2 \).

*If this factor were not a function of \( p \), we'd be worried!* (Why?)
**Inhomogeneous Equations & Problems (§ 20.3.2)**

**Inhomogeneous Equation:**

\[
A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + C u = R(x, y)
\]

Suppose \( u_1(x, y) \) and \( u_2(x, y) \) are solutions.

\(\text{(Show:)}\) \( u_1 + u_2 \) is **not** a solution!

But suppose \( w(x, y) \) is any solution to the homogeneous problem:

\[
A \frac{\partial w}{\partial x} + B \frac{\partial w}{\partial y} + C w = 0.
\]

Then \( u = u_1 + w \) is a solution to 

\(\text{(and so is } u_2 + w)\)

**Example:** \( w = u_2 - u_1 \) would work.

**If** \( w(x, y) \) is the general solution to the homogeneous equation, and \( u_1(x, y) \) is any solution to the inhomogeneous equation, then

\[ u = u_1 + w \]

is the general solution to the inhomogeneous equation.
IT IS POSSIBLE FOR THE **PROBLEM** TO BE **INHOMOGENEOUS** **EVEN WHEN THE EQUATION ISN'T.**

E.g.,

\[
\frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + C u = 0
\]

**WITH INHOMOGENEOUS B.C.** \( u = F(x) \) **ON** \( y = 0 \)

IF \( u_1 \) AND \( u_2 \) **ARE SOLUTIONS TO THE PROBLEM,**

**THE FOLLOWING ARE NOT:**

\[
\begin{align*}
    u &= u_1 + u_2 \quad \rightarrow \quad u(x,0) = 2F(x) \\
    u &= \sin u_1 \quad \rightarrow \quad u(x,0) = \pi F(x) \\
    \text{Etc.}
\end{align*}
\]

**THE BOUNDARY CONDITIONS ARE CONSIDERED HOMOGENEOUS** **IF, FOR ANY** \( u(x,y) \) **THAT SATISFIES THEM,**

\( \lambda u(x,y) \) **ALSO SATISFIES THEM.**

**THE PROBLEM IS HOMOGENEOUS IF WE HAVE BOTH**

(i) **HOMOGENEOUS EQUATION**

(ii) **HOMOGENEOUS B.C.'S**
Example

Solve

\[ y \frac{du}{dx} - x \frac{du}{dy} = 3x \quad (20.18) \]

Subject to \( u(1,0) = 2 \). \( \Leftarrow \) inhomogeneous B.C.

1. First, solve the homogeneous eqn:

\[ y \frac{dw}{dx} - x \frac{dw}{dy} = 0 \]

\[ \frac{dx}{y} = \frac{dy}{-x} \]

\[ \int x \, dx + \int y \, dy = 0 \]

\[ \frac{x^2}{2} + \frac{y^2}{2} = C \]

Gen soln \( w = f(p) \) where \( p = x^2 + y^2 \)

2. Now seek the general solution (without B.C.'s) to the inhomogeneous eq. (20.18).

If we can find a particular soln. \( u(x,y) \), then the general soln. will be \( u = v + w \).

By inspection: \( u(x,y) = -3y \)

Gen soln to (20.18) is thus

\[ u(x,y) = f(x^2 + y^2) - 3y \]
EXAMPLE, CONT.

3 NOW FOR THE B.C.:

\[ U(1,0) = 2 \]
\[ f(1+0) - 0 = 2 \]
\[ f(1) = 2 \]

IF WE CHOOSE, E.G., \( f(p) = 2p \), THEN
\[ f(p) \]
\[ U(x,y) = 2x^2 + 2y^2 - 3y + g(x^2 + y^2) \]
WHERE \( g \) IS ANY FUNCTION WITH \( g(1) = 0 \).

COULD ALSO CHOOSE \( f(p) = 2! \), THEN
\[ U(x,y) = 2 - 3y + g(x^2 + y^2), \quad g(1) = 0. \]

ASSERTION: EACH OF THESE IS THE GENERAL SOLUTION. DO THEY AGREE?

\[ 2x^2 + 2y^2 - 3x + g_1 = 2 - 3x + g_2 \]
\[ g_2 = g_1 + 2p - 2 \]

THIS IS \( g_3(p) \). NOTE THAT \( g(1) = 0 \).