§ 20.3.3  Second-order Equations (Using the same tricks!)

The most general linear, 2nd order PDE on 2 variables would be

\[ A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} + D(x,y) \frac{\partial u}{\partial x} + E(x,y) \frac{\partial u}{\partial y} + F(x,y) u = R(x,y). \]

We begin with a simpler case:

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0 \quad \star \quad (20.20) \]

A, B, C\text{ constants.}

This class of equations includes:

- 1d wave eq.
- 2d Laplace eq.

But does not include:

- Diffusion eq. \require{1st derivatives}
- Schrödinger eq.

Assume soln. of form:

\[ u(x,y) = f(p), \quad p = p(x,y). \]
Now consider the derivatives:

\[
\frac{du}{dx} = f' \frac{dp}{dx}, \quad \frac{du}{dy} = f' \frac{dp}{dy}
\]

\[
\frac{\partial^2 u}{\partial x^2} = f'' \left( \frac{dp}{dx} \right)^2 + f' \frac{\partial^2 p}{\partial x^2}
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = f'' \frac{dp}{dx} \frac{dp}{dy} + f' \frac{\partial^2 p}{\partial x \partial y}
\]

\[
\frac{\partial^2 u}{\partial y^2} = f'' \left( \frac{dp}{dy} \right)^2 + f' \frac{\partial^2 p}{\partial y^2}
\]

These terms are annoying.

When we plug back into \( \Psi \), we hope to be able to eliminate all appearances of \( f \). This seems doomed to fail.

Sleazy trick: choose \( p = ax + by \).

So that the 2nd derivatives go away, then \( \Psi \) becomes:

\[
Af''a^2 + Bf''ab + Cf''b^2 = 0
\]

\[
Aa^2 + Bab + Cb^2 = 0
\]

Note that only the ratio \( b/a \) matters. Divide by \( a^2 \), so

\[
C \left( \frac{b}{a} \right)^2 + B \left( \frac{b}{a} \right) + A = 0
\]

\[
\frac{b}{a} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C} = \{ \lambda_1, \lambda_2 \}
\]
So we have the 2 possibilities

\[ p_1 = x + \lambda_1 y, \quad p_2 = x + \lambda_2 y \]

General soln:

\[ U(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y) \]

Example: 1A wave equation

\[ \frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0 \]

Comparing with \( A \), \( A = 1 \), \( B = 0 \), \( C = \frac{1}{c^2} \)

\[ \lambda_1 = \frac{\sqrt{1/c^2}}{\sqrt{c^2}} = C \]

\[ \lambda_2 = \frac{-\sqrt{1/c^2}}{\sqrt{c^2}} = -C \]

So

\[ U(x, t) = f(x - ct) + g(x + ct) \]

Which is what we should expect.
Example

Now, try Laplace's equation:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

\[A = C = 1, \quad B = 0\]

\[
\lambda = \frac{b}{a} = \frac{\pm \sqrt{-1}}{2} = \pm i
\]

\[U = f(x+iy) + g(x-iy)\]

Which is a result familiar to students of complex analysis.

Note that:

\[
\nabla^2 \phi(z) = 0, \quad \phi(z) = \text{Re } f + \text{Im } f
\]

\[
\Rightarrow \text{Re } (\nabla^2 \phi(z)) = \text{Im } (\nabla^2 \phi(z)) = 0
\]

\[
\Rightarrow \nabla^2 (\text{Re } f(z)) = \nabla^2 (\text{Im } f(z)) = 0
\]

So each \(f(x+iy)\) represents 2 real solutions in 2D.
ASIDE: CLASSIFICATION OF PDE'S OF FORM \( A \phi \) \n
\[ B^2 > 4AC \quad \rightarrow \quad \text{HYPERBOLIC EQUATIONS} \]

\[ B^2 < 4AC \quad \rightarrow \quad \text{ELLIPTIC} \]

\[ B^2 = 4AC \quad \rightarrow \quad \text{PARABOLIC} \]

FOR \( A, B, C \) REAL, HYPERBOLIC EQUATIONS HAVE SOLUTIONS LIKE \( f(x + \alpha y), \alpha \in \mathbb{R} \).

ELLIPTIC EQUATIONS HAVE SOLUTIONS LIKE \( f(x + i\beta y), \beta \in \mathbb{R} \).

Q: WHAT TYPE IS THE WAVE EQ?
A: HYPERBOLIC

Q: WHAT TYPE IS LAPLACE'S EQ?
A: ELLIPTIC
PARABOLIC EQUATIONS (§ 20.3.3, STARTING P. 690)

FROGGING APPROACH DOESN'T QUITE WORK FOR
PARABOLIC EQUATIONS \( B^2 = 4AC \); WE WOULD
GET ONLY

\[
U(x,y) = f(x - \frac{B}{2C} y)
\]

THIS IS OK, EXCEPT THAT WE

\* SHOULD EXPECT 2 COMBINATIONS, SINCE
IT IS A 2ND ORDER EQUATION.*
To find the "rest" of the soln, try

\[ u(x, y) = h(x, y) \, g \left( x - \frac{B}{2c} y \right) \]

Derivatives are:

\[ \frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} \, g + h \, g' \]

\[ \frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} \, g - \frac{B}{2c} \, h \, g' \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 h}{\partial x^2} \, g + 2 \frac{\partial h}{\partial x} \, g' + h \, g'' \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial h}{\partial x} \, g - \frac{B}{2c} \frac{\partial h}{\partial x} \, g' + \frac{\partial h}{\partial y} \, g' - \frac{B}{2c} \, h \, g'' \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 h}{\partial y^2} \, g - \frac{B}{c} \frac{\partial h}{\partial y} \, g' + \frac{B^2}{4c^2} \, h \, g'' \]

Sub into A:

\[ A \, g \left( \frac{\partial h}{\partial x} \right)^2 + 2A \, g' \frac{\partial h}{\partial x} + A \, h \, g'' \]

\[ + B \, g \frac{\partial h}{\partial x \partial y} - \frac{B^2}{2c} \frac{\partial h}{\partial x} \, g' + \frac{B}{2c} \frac{\partial h}{\partial y} \, g' - \frac{B^2}{2c} \, h \, g'' \]

\[ + C \, g \frac{\partial h}{\partial y} - B \, g' \frac{\partial h}{\partial y} + \frac{B^2}{4c} \, h \, g'' \]

= 0
The case we're interested in is

\[ B^2 - 4AC = 0 \]

i.e. \( A = \frac{B^2}{4C} \)

\[ \Rightarrow \; g'' \text{ terms cancel} \]

\[ \Rightarrow \; g' \text{ terms cancel} \]

We are left with (as usual!)

\[
\left( A \frac{\partial^2 h}{\partial x^2} + B \frac{\partial h}{\partial x \partial y} + C \frac{\partial^2 h}{\partial y^2} \right) g = 0
\]

\[ \Rightarrow \; h(x, y) \text{ must be a particular soln. to } A \]

Our general soln. will be:

\[ u(x, y) = f(x - \frac{B}{2C} y) + h \cdot g(x - \frac{B}{2C} y) \]

It may be tempting to choose \( h = x - \frac{B}{2C} y \), but this won't work. (why?)

Instead choose \( h = x \).

Q: Is \( h = x \) a soln. to \( A \) ?
A: Yes.

Q: Is \( h = x \) a soln. for the elliptic and hyperbolic cases too?
A: Yes.

Q: Is \( h = x \) included in our "general" solns.?
A: Yes. \( h = x = \frac{1}{2}(P_1 + P_2) \), e.g. for elliptic case.
RESULT:

For \( b^2 = 4ac \), (Parabolic)

\[ u(x,y) = f(x - \frac{b}{2c} y) + x g(x - \frac{R}{2c} y) \]

EXAMPLE (From book)

\[ \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \]

B.C.: \[ u(0,y) = 0 \]
\[ u(x,1) = x^2 \]

SOLN: - Parabolic, \( p = x - y \)
- \( f(p) = 0 \) \( \Rightarrow \) BC CONSEQUENCES
- \( g(p) = p + 1 \)

\[ u(x,y) = x(p+1) = x^2 - xy + x. \]