§ 20.6.1 — CHARACTERISTICS FOR 1ST ORDER EQ'S

CONSIDER THE GENERAL 1ST ORDER PDE ON 2 VAR'S:

\[ A(x,y) \frac{du}{dx} + B(x,y) \frac{du}{dy} = F(x,y,u) \]

NOTE HOW U HAS BEEN MOVED TO THE RHS.

Q: WHAT KIND OF BOUNDARY CONDITIONS HAVE, IN GENERAL, BEEN SUFFICIENT TO SPECIFY A PARTICULAR SOLN? E.G.:

\( A \) \hspace{1cm} \( B \)
\[ u(x,0) = \varphi(x). \]
\[ u(1,3) = 4. \]

A: \( A \) YES, \( B \) NO.

LET'S TRY TO MAKE UP A MORE GENERALIZED VERSION OF \( A \):

LET \( u(x,y) = \varphi(s) \)

ON PARAMETRIC CURVE
\[ x = x(s), \quad y = y(s). \]

NOW, CONSIDER

\[ \frac{du}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = \frac{d\varphi}{ds}. \]
We can rewrite \( \Phi \) and \( \Psi \) as:

\[
\begin{pmatrix}
A(x, y) & B(x, y) \\
\frac{dx}{ds} & \frac{dy}{ds}
\end{pmatrix}
\begin{pmatrix}
\frac{du}{dx} \\
\frac{du}{dy}
\end{pmatrix}
= \begin{pmatrix}
F(x, y, u) \\
\frac{d\Phi}{ds}
\end{pmatrix}
\]

This can be solved using Cramer's Rule (p. 299) to give us \( \frac{dx}{du} \) and \( \frac{dy}{du} \) on the boundary curve \((x(s), y(s))\). (We won't do this.)

Notice: Cramer's Rule has the determinant of the matrix in the denominator. There is no solution if

\[
\begin{vmatrix}
A & B \\
x' & y'
\end{vmatrix} = 0.
\]

So the solutions to the above equation are boundary curves that don't work:

\[
A \frac{dy}{ds} - B \frac{dx}{ds} = 0
\]

or

\[
\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)} \tag{20.41}
\]

This should look familiar — it is the equation for surfaces of constant \( p(x, y) \), where

\[
U(x, y) = h(x, y) f(p).
\]
**Solutions to 20.41 Look Like**

\[ p(x,y) = \text{const.} \]

These curves of constant \( p \) are called **characteristics**.

**Implications:**

- **Setting the Value of** \( u(x,y) \) **at one point on a characteristic curve determines** \( u \) **everywhere along that curve, because it fixes** \( f(p) \) **for that particular value of** \( p \).

- **If the boundary curve runs tangent to the characteristics at any point, then** \( u \) **is overdetermined because it fixes** \( f(p) \) **for a certain** \( p \) **more than once**.

- **Similarly, if the boundary crosses a characteristic more than once, \( u \) may be overdetermined.**

**Interpretation:**

The characteristics map the paths along which information flows from a boundary condition.
EXAMPLE (the book gives a different, equally good example)

1D ADVECTION EQUATION:

\[ \frac{du}{dt} = -u(x,t) \frac{du}{dx} \] (Discuss physically)

For simplicity, take \( u = \text{const.} \),

\[ \frac{du}{dt} + u \frac{du}{dx} = 0 \]

Assume gen soln \( u(x,t) = f(p(x,t)) \).

Find \( p \):

\[ \frac{dt}{1} = \frac{dx}{u} \]

\[ t = \frac{x}{u} + \text{const.} \]

\[ p = x - u \cdot t \; ; \; \; u = f(x - u \cdot t) \]

CHARACTERISTICS:

Lines of slope \( \frac{dt}{dx} = \frac{1}{u} \).
CASE I

Suppose we have the boundary condition

\[ u(x, 0) = e^{-x^2} \]

- Boundary curve \( t = 0 \)
- BC intersects all characteristic curves.
- BC intersects each characteristic once.

Thus, \( u(x, t) \) should be fully determined everywhere in the plane.

Solution:

\[ f(x - vt) = e^{-x^2} \]
\[ f(x) = e^{-x^2} \]

Thus \( u(x, t) = f(x - vt) = e^{-\left(x - vt\right)^2} \)
For each of the following cases,
(a) Sketch the boundary curve
(b) Specify where \( u(x, t) \) is known
   (and give the explicit solution)
(c) Specify where \( u(x, t) \) is unknown
(d) Specify where \( u(x, t) \) is overdetermined

Case II: \( u(x, 0) = \cos x \) for \( -\frac{\pi}{2} < x < \frac{\pi}{2} \)
   (I do this one)

Case III: \( u(0, t) = u(1, t) = x \)

Case IV: \( u(x, t) = 1 \) on \( x^2 + y^2 = 1 \).

Case V: \( u(x, t) = x^2 \)

Case II:

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(a) B.C.
(b) Known in shaded region
(c) Unknown outside shaded region
(d) Nowhere overdetermined