MINOR COMMENT ON STURM-LIOUVILLE FORM:

\[-\frac{1}{\rho(x)} \left[ p(x) \, u'' + p'(x) \, u' + q(x) \, u \right] = \lambda \, u\]

*Can also be written*

\[-\frac{1}{\rho(x)} \left[ \frac{d}{dx} \left( p(x) \, u' \right) + q(x) \, u \right] = \lambda \, u.\]

*This is the key thing behind SL-form.*

WE HAVE BEEN IN §21.3, PERFORMING SOLUTIONS OF

\[\nabla^2 u = 0\]

IN PLANE POLAR, CYLINDRICAL POLAR, AND (TODAY)
SFERICAL POLAR COORDINATES.
LAPLACE'S Eq. IN SPHERICAL COORDINATES

$$\nabla^2 u = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

As usual, assume

$$u(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

so, multiplying $A$ by $r^2/u$,

$$\frac{1}{R} \frac{d}{dr} \left( r^2 R' \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \Theta' \right) + \frac{1}{\Phi \sin^2 \theta} \Phi'' = 0$$

Function of $r$ only

Function of $\theta, \phi$ only

$$\Rightarrow \quad \frac{1}{R} \frac{d}{dr} \left( r^2 R' \right) = \lambda$$

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \Theta' \right) + \frac{1}{\Phi \sin^2 \theta} \Phi'' = -\lambda \quad (21.41)$$

$$r^2 R'' + 2r R' - \lambda R = 0$$

DEFINITELY A STURM-LIOUVILLE EQUATION, BUT NOT IN TABLE 17.1.
This one is again susceptible to the change of variables:

\[ r = e^t \]

\[ \frac{dR}{dt} = R' \frac{dr}{dt} = R' r \]

\[ \frac{d^2R}{dt^2} = r \frac{dR'}{dt} + R' r = r^2 R'' + r R' \]

\( \text{(Notation: } R' = \frac{dR}{dr} \text{)} \)

And so our original eq. may be written

\[ \frac{d^2R}{dt^2} + \frac{dR}{dt} - \lambda R = 0 \]

Since we have constant coeff's, try

\[ R = e^{\lambda t} \]

\[ \Rightarrow \frac{dR}{dt} = \lambda R, \quad \frac{d^2R}{dt^2} = \lambda^2 R \]

So

\[ \lambda^2 + \lambda - \lambda = 0 \quad \Rightarrow \quad \lambda = \lambda(l+1) \]

\[ \frac{\lambda}{2} = \frac{-1 \pm \sqrt{1 + 4 \lambda}}{2} = \frac{-1 \pm \sqrt{1 + 4 \lambda(l+1)}}{2} \]

\[ \lambda = \frac{-1 \pm (2l+1)}{2} = \left\{ \begin{array}{l} \lambda \\ -(l+1) \end{array} \right. \]

In other words, for a particular \( \lambda = \lambda(l+1) \), we may have soln's \( e^{\lambda t} \) or \( e^{-(l+1)t} \).
We therefore have

\[ R = A e^{\lambda t} + B e^{-(\lambda+1)t} \]

\[ R(r) = Ar^\lambda + Br^{-(\lambda+1)} \]

And 21.41 (multiplied by \( \sin^2 \theta \), with \( \lambda \rightarrow \lambda(\lambda+1) \)) becomes

\[ \sin \theta \frac{d}{d\theta} \left( \sin \theta \Phi' \right) + \lambda(\lambda+1) \sin^2 \theta \Phi + \Phi = 0 \]

\( \phi \) only

\( \theta \) only

Let \( \frac{\Phi''}{\Phi} = -m^2 \), \( \Phi = C \cos m\phi + D \sin m\phi \)

And then the \( \theta \) equation is

\[ \sin \theta \frac{d}{d\theta} \left( \sin \theta \Phi' \right) + \lambda(\lambda+1) \sin^2 \theta \Phi = m^2 \Phi \]  

(21.44)

Clearly a Sturm-Liouville Eq.,

But Table 17.1 has no \( p(x) = \sin x \),

We can get rid of the trig functions with the following change of variables:

Let \( \mu = \cos \theta \), \( \Phi(\theta) = M(\mu) \)

So that

\[ \frac{d}{d\theta} = \frac{d}{d\mu} \frac{d}{d\mu} = -\sin \theta \frac{d}{d\mu} = -\sqrt{1-M^2} \frac{d}{d\mu} \]

Now 21.44 becomes:

\[ -(1-M^2) \frac{d}{dM} \left( - (1-M^2) M' \right) + \lambda(\lambda+1) (1-M^2) M = m^2 M \]
\[(1-m^2) \left[ \frac{d}{dm} \left( (1-m^2) M' \right) + \lambda (l+1) M \right] = m^2 M \quad \text{!!} \]

This is a Sturm-Liouville equation... but no equivalent in Table 17.1.

We can rearrange differently, letting \( \lambda (l+1) \) be the eigenvalue:

\[- \left[ \frac{d}{dm} \left( (1-m^2) M' \right) - \frac{m^2}{(1-m^2)} M \right] = \lambda (l+1) M \]

Again a Sturm-Liouville equation, now recognizable as the associated Legendre equation. (§ 18.2)

**Solutions:** \( M = P^m_l (m) \), \( Q^m_l (m) \)

\( \Theta = E P^m_l (\cos \theta) + F Q^m_l (\cos \theta) \)

So we have

\[
U(r, \theta, \phi) = \left( A r^l + B r^{-(l+1)} \right) \left( C \cos m \phi + D \sin m \phi \right) \\
\times \left( E P^m_l (\cos \theta) + F Q^m_l (\cos \theta) \right) \quad (21.49)
\]

We require \( m \in \mathbb{I} \), \( l \in \mathbb{I} \), and

\[ |m| \leq l \]

These considerations arise from forming finite polynomial solutions of the associated Legendre equation.
Summary of the Integer-λ Legendre Functions: (§18.1)

For \( m = 0 \) (the case treated, e.g., in Ch. 3 of Griffiths), we have azimuthal symmetry. This gives the Legendre equation,

\[
\frac{d}{d\mu} \left( (1-\mu^2)M' \right) + \lambda(\lambda+1)M = 0
\]

for which solutions are

\[
M = aP_\lambda(\mu) + bQ_\lambda(\mu)
\]

\[
P_\lambda(x) = \frac{1}{2^\lambda\lambda!} \left( \frac{d}{dx} \right)^\lambda (x^2-1)^\lambda \quad \text{(Rodrigues Formula)}
\]

\[
P_0(x) = 1 \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)
\]
\[
P_1(x) = x \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)
\]
\[
P_2(x) = \frac{(3x^2 - 1)}{2} \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)
\]

Etc.

There exists another set of solutions, Legendre functions of the "second kind:\

\[
Q_{\lambda}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)
\]

\[
Q_0(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) - 1
\]

Recurrence relation:

\[
(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \quad (18.27)
\]

\[
(n+1)Q_{n+1} = (2n+1)xQ_n - nQ_{n-1}
\]
Figure 18.1  The first four Legendre polynomials.

Figure 18.2  The first three Legendre functions of the second kind.
Figure 18.1  The first four Legendre polynomials.

Figure 18.2  The first three Legendre functions of the second kind.
Since in physical applications the argument is $\cos \theta = x$, the usual domain of the Legendre functions is $-1 \leq x \leq 1$.

Note that $P_n(\cos \theta)$ blows up at $\theta = 0, \pi$.

Thus, $P_n$'s are not used in cases where the z-axis is in the domain. That is why Griffiths ignores them!
ASSOCIATED LEGENDRE FUNCTIONS

WHEN WE DON'T HAVE AXISYMMETRY, WE MUST DEAL WITH THE FULL EQUATION, \((m \neq 0)\)

\[
\frac{d}{dm} \left( (1-m^2) M' \right) - \frac{m^2}{(1-m^2)} M + \lambda (l+1) M = 0.
\]

SOLUTIONS \( M = a P^m_l (m) + b Q^m_l (m) \)

WHERE

\[
\begin{align*}
P^m_l (x) &= (1-x^2)^{m/2} \left( \frac{d}{dx} \right)^m P_l (x) \\
Q^m_l (x) &= (1-x^2)^{m/2} \left( \frac{d}{dx} \right)^m Q_l (x)
\end{align*}
\]

(18.32)

NOTE THAT \( m^2 \) (NOT \( m \)) APPEARS IN THE ASSOC. LEGENDRE EQUATION. THIS IMPLIES THAT

\( P^m_l (x) \propto P^m_l (x) \) (SAME FOR \( Q \)).

SINCE \( P_l \) IS A POLYNOMIAL OF ORDER \( l \),

\( P^m_l (x) = 0 \) FOR \(|m| > l\).

ORTHOGONALITY:

\[
\int_{-1}^{1} P^m_l (x) P^m_k (x) = 0 \text{ for } l \neq k
\]

NORMALIZATION:

\[
\int_{-1}^{1} \left[ P^m_l (x) \right]^2 dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}
\]

\( P^m_l \)S ARE A BASIS FOR ANY \( m \)!
Spherical Harmonics

\[ Y_{\ell}^m(\theta, \phi) = (-1)^m \left[ \frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!} \right]^{1/2} P_{\ell}^m(\cos \theta) \, e^{im\phi} \]

and \[ Y_{\ell}^{-m} = (-1)^m Y_{\ell}^m \]

These are properly normalized!

First few \(Y_{\ell}^m\)'s are tabulated in §18.45.

\[ \int_{-1}^{1} d(\cos \theta) \int_{0}^{2\pi} d\phi \ Y_{\ell}^m \ast Y_{\ell'}^{m'} = S_{\ell \ell'} S_{m m'} \]

Of course, \[ d(\cos \theta) = -\sin \theta \, d\theta \]

so \[ \int_{-1}^{1} d(\cos \theta) = -\int_{0}^{\pi} \sin \theta \, d\theta = \int_{0}^{\pi} \sin \theta \, d\theta \]

Since \[ d\Omega = \sin \theta \, d\theta \, d\phi \], we have

\[ \int Y_{\ell}^m \ast Y_{\ell'}^{m'} \, d\Omega = S_{\ell \ell'} S_{m m'} \]

Our solution to \( \nabla^2 u = 0 \) may now be expressed compactly:

\[ U(r, \theta, \phi) = \sum_{\ell, m} \left( A_{\ell m} r^\ell + B_{\ell m} r^{-(\ell + 1)} \right) Y_{\ell m}(\theta, \phi) \]

\[ \ell = 0, 1, 2, \ldots \]

\[ m = -\ell, -\ell + 1, \ldots, \ell \]