INHOMOGENEOUS PDES (§ 21.5) — GREEN'S FUNCTIONS

Suppose we have the problem

\[ \nabla \cdot \mathbf{\phi}(\mathbf{r}) = \mathbf{p}(\mathbf{r}) \]

Could have included \( t \), or any list of independent variables!

Subject to \underline{Homogeneous B.C.'s} (for now).

Suppose we know \( G(\mathbf{r}, \mathbf{r}_0) \) such that

\[ \nabla \cdot G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0), \]

where \( G \) satisfies our homogeneous B.C.'s.

We can express

\[ \mathbf{p}(\mathbf{r}) = \int_D \mathbf{p}(\mathbf{r}_0) \, \delta(\mathbf{r} - \mathbf{r}_0) \, d\mathbf{V}_0 \]

\[ = \int_D \mathbf{p}(\mathbf{r}_0) \left[ \nabla \cdot G(\mathbf{r}, \mathbf{r}_0) \right] d\mathbf{V}_0, \]

\[ \mathbf{p}(\mathbf{r}) = \nabla \left( \int_D \mathbf{p}(\mathbf{r}_0) \, G(\mathbf{r}, \mathbf{r}_0) \, d\mathbf{V}_0 \right) \]

Satisfies \( \mathcal{A} \) (and B.C.'s!); \( \mathcal{A} = \mathbf{\phi}(\mathbf{r}) \)

\[ \therefore \mathbf{\phi}(\mathbf{r}) = \int_D \mathbf{p}(\mathbf{r}_0) \, G(\mathbf{r}, \mathbf{r}_0) \, d\mathbf{V}_0 \]

**The main question is how to find \( G \)!!**

- Ansatz (guess)
- Fourier Transform (HW 7, #2)
- Method of Images
- Eigenfunction Approach (tedious in practice)
**Example**

Suppose $\nabla^2 u = \rho(r)$, Domain $x > 0$.

And $u = 0$ on $x = 0$.

Find $G(r, r_0)$ (Method of Images!)

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Q: What if the B.C.'s were: $u = 0$ on $xy = 0$,

A: For a charge at $x_0, y_0$,

Place Image Charges at:

- $-x_0, y_0$
- $x_0, -y_0$
- $-x_0, -y_0$

(WRITE THE G.F.!)
EIGENFUNCTION APPROACH: (§21.5.1)

We wish to solve

\[ \mathcal{L} \psi_n(\vec{r}) = \lambda_n \psi_n(\vec{r}), \quad \langle \psi_n | \psi_m \rangle = \delta_{mn} \]

Suppose \( \mathcal{L} \), with our B.C.'s, is HERMITIAN.

Since the \( \psi_n \) form an orthonormal basis, we can write

\[ \delta(\vec{r} - \vec{r}_0) = \sum_n c_n \psi_n(\vec{r}) \]

We can find the \( c_n \) by the usual trick: (times \( \psi_n^*(\vec{r}) \); \( \int dV \))

\[ \int dV \psi_n^*(\vec{r}) \delta(\vec{r} - \vec{r}_0) = c_n \int dV \psi_n^* \psi_n \]

\[ \Rightarrow \quad c_n = \psi_n^*(\vec{r}_0). \]

So,

\[ \delta(\vec{r} - \vec{r}_0) = \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}). \]

Let \( G(\vec{r}, \vec{r}_0) = \sum_n a_n \psi_n(\vec{r}) \)

Then \( \mathcal{L} \) becomes:

\[ \mathcal{L} \left( \sum_n a_n \psi_n(\vec{r}) \right) = \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) \]

\[ \Rightarrow \quad a_n \lambda_n = \psi_n^*(\vec{r}_0) \]

\[ \Rightarrow \quad G(\vec{r}, \vec{r}_0) = \sum_n \frac{1}{\lambda_n} \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) \quad (2.1.79) \]
THINGS TO NOTE:

- \( G(r, r_0) = G^*(r_0, r) \)
- SAME APPROACH WORKS FOR ODE GREEN'S FUNCTIONS (SEE CH.17).
- IF \( D \) IS AN OPERATOR WITH CONSTANT COEFFICIENTS, THEN THE \( \psi_m \) WILL BE COMPLEX EXPONENTIALS, AND (21.79) IS JUST THE FOURIER TRANSFORM SOLUTION OF THE PROBLEM. (SEE HW 7)
Example

$$\nabla^2 \Phi = -\rho(\vec{r})/\varepsilon_0$$

Inside a conductive, grounded (\(\Phi = 0\)) cube:

![Diagram of a cube with labels x, y, z, d, d, d]

The B.C. is homogeneous Dirichlet.

Build G.F. from eigenfunctions:

$$\nabla^2 \phi_n(\vec{r}) = \lambda_n \phi_n(\vec{r})$$

$$G(\vec{r}, \vec{r}_0) = \sum_n \frac{\phi_n^*(\vec{r}_0) \phi_n(\vec{r})}{\lambda_n} \quad (21.79)$$

Q: What must be true of \(\nabla^2\) for this to work?

A: Hermitian over given domain & B.C.'s
Is $\nabla^2$ Hermitian in this case?

For $f(\vec{r})$, $g(\vec{r})$ conforming to our B.C.'s, show:

$$\int_V f^*(\vec{r}) \nabla^2 g(\vec{r}) \, dV = \left[ \int_V g^*(\vec{r}) \nabla^2 f(\vec{r}) \, dV \right]^*$$

Note that $(\nabla^2)^* = \nabla^2$, and let $h(\vec{r}) = f^*(\vec{r})$.

Now we want to prove:

$$\iff \int_V h(\vec{r}) \nabla^2 g(\vec{r}) \, dV = \int_V g(\vec{r}) \nabla^2 h(\vec{r}) \, dV$$

Consider Green's Second Theorem: (we'll use it again soon!)

$$\int_V (h \nabla^2 g - g \nabla^2 h) \, dV = \int_{\partial V} (h \nabla g - g \nabla h) \cdot d\vec{A} = 0$$

Since $h = g = 0$ on $S'$.

$$\therefore \int_V h \nabla^2 g \, dV = \int_V g \nabla^2 h \, dV$$

Q.E.D.

Note $\nabla^2$ is Hermitian on homogeneous Dirichlet and/or homogeneous Neumann B.C.'s:

$$\int_{\partial V} (h \nabla g - g \nabla h) \cdot d\vec{A} = \int_{\partial V} [h(\nabla \cdot \vec{v}) - g \cdot (\nabla \cdot \vec{v})] \, d\vec{A}$$

Normal derivatives
Now, the eigenvalue problem:

\[ \nabla^2 \varphi = \lambda \varphi, \quad \varphi = 0 \quad \text{at} \quad xyz(x-d)(y-d)(z-d) = 0 \]

Solutions:

\[ \varphi = A \sin \left( \frac{l \pi x}{d} \right) \sin \left( \frac{m \pi y}{d} \right) \sin \left( \frac{n \pi z}{d} \right) \]

\[ \lambda = - \left( \frac{\pi^2}{d_x^2} + \frac{\pi^2}{d_y^2} + \frac{\pi^2}{d_z^2} \right) \]

Boundary conditions:

\[ \varphi = 0 \quad \text{at} \quad x = d \quad \Rightarrow \quad \frac{l \pi x}{d} = \pi l, \quad l = 1, 2, 3 \]

And similarly in y, z.

So

\[ \varphi_{lmn} = A \sin \left( \frac{l \pi x}{d} \right) \sin \left( \frac{m \pi y}{d} \right) \sin \left( \frac{n \pi z}{d} \right) \]

Normalization (never forget this step!)

\[ \iiint_0^d \varphi \cdot \varphi \, dx \, dy \, dz = 1 \]

\[ A^2 \left( \frac{1}{d} \right)^3 = 1 \quad \Rightarrow \quad A = \left( \frac{2}{d} \right)^{\frac{3}{2}} \]

\[ \varphi_{lmn} = \left( \frac{2}{d} \right)^{\frac{3}{2}} \sin \left( \frac{l \pi x}{d} \right) \sin \left( \frac{m \pi y}{d} \right) \sin \left( \frac{n \pi z}{d} \right) \]

\[ l, m, n \in \{ 1, 2, 3, \ldots \} \]

\[ \lambda_{lmn} = -\frac{l^2}{d_x^2} = -\frac{\pi^2}{d_x^2} \left( l^2 + m^2 + n^2 \right) \]
Then,

\[ G(\vec{r}, \vec{r}_0) = \sum_{l,m,n} \frac{- \phi_{l,m,n}(\vec{r}) \phi_{l,m,n}(\vec{r}_0)}{\frac{\pi^2}{d^2} (l^2 + m^2 + n^2)} \]

\[ = \sum_{l,m,n} \frac{-8 \sin(l\pi x/d) \sin(l\pi y/d) \sin(l\pi z/d)}{\pi^2 d (l^2 + m^2 + n^2)} \sin(l\pi x/d) \sin(l\pi y/d) \sin(l\pi z/d). \]

Q: How many terms does it take to converge to \( \sim 1\% \)?

A: \( \approx 10^3 \) terms: \( l = 1 \ldots 10, \ m = 1 \ldots 10, \ n = 1 \ldots 10 \)

Q: For 1\% accuracy on a 100 x 100 x 100 grid, how many times would you have to call the \( \sin \) function?

A: 6 calls \( \times 10^3 \) terms \( \times 10^6 \) grid points.

\[ = 6 \times 10^9. \text{ Ouch!} \]

Still, the cost of a gigaflop is small, so this is not so bad for numerical use.
Q: Given the slow convergence of the eigenfunction expansion, is it good for anything?

A: Sure. If \( \rho(\vec{r}) \) is simple enough in terms of \( \phi_{l,m,n} \), then it's great. For example...

Suppose: \( \rho(\vec{r}) = \rho_0 \sin(\pi x/d) \sin(\pi y/d) \sin(2\pi z/d) \)

\[\begin{align*}
\text{Cloud of } \varphi \text{ charge} \\
\text{Cloud of } \bar{\varphi} \text{ charge}
\end{align*}\]

Q: Now, what is \( \bar{\varphi}(\vec{r}) \)?

A: Begin by writing

\[-\frac{\rho_c(\vec{r})}{\epsilon_0} = -\rho_0 \left( \frac{d}{2} \right)^{3/2} \phi_{112}(\vec{r}).\]

So,

\[\bar{\varphi}(\vec{r}) = + \sum_{l,m,n} \frac{\phi_{l,m,n}(\vec{r}) \phi_{l,m,n}^*(\vec{r}_0)}{\left( \frac{l^2}{a^2} \right) \left( \frac{l^2 + m^2 + n^2}{d^2} \right)} \left( \frac{d}{2} \right)^{3/2} \rho_0 \epsilon_0 \ dV_0\]

Since \( \langle \phi_{l,m,n} | \phi_{l,m,n}^* \rangle = \delta_{ll'} \delta_{mm'} \delta_{nn'} \), the sum goes away and \( l, m, n = 1, 1, 2 \ldots \)
\[ \Phi (\vec{r}) = \frac{d^2}{6\pi^2} \frac{P}{\varepsilon_0} \sin \left( \frac{\pi x}{d} \right) \sin \left( \frac{\pi y}{d} \right) \sin \left( \frac{2\pi z}{d} \right) \]

Now we see the power of the eigenvalue approach.

Next time:

Inhomogeneous B.C.'s