Comment on §21.5.4

In the Neumann problem,

\[ \nabla^2 u = \rho(\vec{r}) \text{ on } V \text{ bounded by } S, \]
\[ \frac{\partial}{\partial n} u(\vec{r}) = \psi(\vec{r}) \text{ on } S. \]

We have here a possible inconsistency:

\[ \int_V \rho(\vec{r}) \, dV = \int_V \nabla \cdot (\nabla u) \, dV \]
\[ = \oint_S \hat{n} \cdot (\nabla u) \, dA \quad (\text{Gauss's Thm}) \]

\[ \oint_S \rho(\vec{r}) \, dA = \oint_S \psi(\vec{r}) \, dA \]

Requirement for consistency.

Thus, \( \psi(\vec{r}) \) cannot be quite arbitrary!

This becomes a serious concern when we try to construct a Green's fn.
Our usual approach, starting from Eq. (21.84), would be:

\[ U(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}_0) \rho(\mathbf{r}_0) dV, \]

\[ + \oint_S \left[ U(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n_0} - G(\mathbf{r}, \mathbf{r}_0) \frac{\partial U(\mathbf{r}_0)}{\partial n_0} \right] dA_0, \]

Choose \( G(\mathbf{r}, \mathbf{r}_0) \) satisfying

\[ \nabla^2 G = \delta(\mathbf{r}, \mathbf{r}_0) \]

And let \( \frac{\partial G}{\partial n_0} = 0 \) on \( S \).

What's the problem here?

→ If \( \nabla^2 G = \nabla \cdot (\nabla G) = \delta(\mathbf{r}, \mathbf{r}_0) \)

Then \( \int_V \nabla^2 G \, dV = 1 \)

\[ 1 = \oint_S \hat{n} \cdot \nabla G \, dA = \oint_S \frac{\partial G}{\partial n} \, dA \]

\[ 1 = 0 \quad \text{oops!} \]
RHB proposes the following solution:

\[
\frac{\partial G}{\partial n} \bigg|_{\text{sc}} = \frac{1}{A}, \quad A = \oint_{\text{sc}} \, dA
\]

\[A:\quad \int_{V} \nabla^{2} G \, dv = 1 = \oint_{\text{sc}} \frac{\partial G}{\partial n} \, dA
\]

\[= \frac{1}{A} \oint_{\text{sc}} G \, dA = 1 \checkmark
\]

**Comment:** It can be very difficult to find such a Green's function. However, a nice example is given.

Now, Eq. (21.84) becomes:

\[
U(F) = \int_{V} G \rho(F_{0}) \, dv + \oint_{\text{sc}} \left[ \frac{U(F_{0})}{A} - G \Psi(F_{0}) \right] \, dA
\]

\[= \int_{V} G \rho(F_{0}) \, dv + \langle u \rangle_{\text{sc}} - \oint_{\text{sc}} G \Psi(F_{0}) \, dA .
\]

Note that this is an arbitrary constant.

I will not emphasize this approach. The trick isn't generically useful, and B.C.'s of this form are not so common (imposing \( \frac{\partial u}{\partial n} \) is unphysical!)
\[ \nabla^2 \Phi = -\frac{\rho(\vec{r})}{\epsilon_0} \]

Solve for \( x > 0 \)

(\( x < 0 \) is symmetric)

**Physics tells us:**

\[ \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}' - \vec{r}|} dV' \]

\[ = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dy_0 \int_{-\infty}^{\infty} dz_0 \frac{\sigma(y_0, z_0)}{\sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}} \]

We can use this as a test of our ability to handle Neumann B.C.'s:

\[ \oint_{S_p} \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \]

\[ 2E_n(y, z)A = \left( \frac{\sigma(y,z)}{\epsilon_0} \right) A \]

\[ E_n(x, z) = \frac{\sigma(x,z)}{2\epsilon_0} = \frac{\partial \Phi}{\partial n} \]
GREEN'S FUNCTION: USE POSITIVE MIRROR CHARGE (WHY?)

\[ \nabla \cdot G = S (\vec{r} - \vec{r}_0) \]

\[ G = F + H \quad \text{WHY BOTH TERMS} < 0 \? \quad \text{(DISCUSS)} \]

\[ = -\frac{1}{4\pi|\vec{r} - \vec{r}_0|} + \frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \]

Q: IS THERE ANY SELF-CONSISTENCY ISSUE?
A: NO, IMPLIED DIRICHLET B.C. \( \Phi \to 0 \) AS \( |x| \to \infty \).

NOW, EQ. 21.84 BECOMES:

\[ u(\vec{r}) = \int_B G(x', \vec{r}) \, d\vec{r}_0 - \frac{1}{2} \int_B G(x, \vec{r}_0) \frac{\partial u(x)}{\partial n} \, d\vec{r}_0 \]

\[ = -\int_B \frac{G(x, \vec{r}_0)}{2e_0} \left[ \frac{-2}{4\pi \sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}} \right] \, d\vec{r}_0 \]

\[ = \frac{+1}{4\pi e_0} \left( \int_{-\infty}^{\infty} d\bar{z} \int_{-\infty}^{\infty} dy \right) \frac{5(\bar{y}, \bar{z})}{\sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}} \]

\[ \uparrow \]

LOOKS LIKE PHYSICS!