COMPLEX ANALYSIS (RHB CH.24)

PRESENTATION HERE FOLLOWS THE TEXT CLOSELY. RHB ITSELF GLOSSES OVER MANY MATHEMATICAL DETAILS; IF THIS BOthers YOU, YOU MAY WISH TO READ ALONG IN A COMPLEX ANALYSIS TEXT. I HAVE FOUND CHERCHILL & WARD* TO BE READABLE.

FUNCTIONS OF A COMPLEX VARIABLE (§24.1)

\[ z = x + iy \quad (z \in \mathbb{C}; \quad x, y \in \mathbb{R}) \]

\[ f(z) = u(x,y) + iv(x,y) \quad (u, v \in \mathbb{R}) \]

\[ f(z) \text{ is } \underline{\text{analytic}} \text{ on domain } \Omega \text{ (subset of } \mathbb{C}) \text{ if} \]

\[ \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \]

EXISTS AND IS INDEPENDENT OF THE DIRECTION FROM WHICH \( \Delta z \) GOES TO ZERO, \( \forall z \in \mathbb{R} \).

EXAMPLE

Show that \( f(z) = \text{Re}(z) \) is not analytic.

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

Now let \( \Delta z = \Delta x + i \Delta y \).

\[
f' = \lim_{\Delta z \to 0} \frac{X + \Delta X - X}{\Delta X + i \Delta Y} = \frac{\Delta X}{\Delta X + i \Delta Y}
\]

The answer depends on our path of approach to \( \Delta z = 0 \).

Case I: \( \Delta X \to 0 \) first (\( \Delta z \to 0 \) along real axis)

\[
\Rightarrow f' \to \frac{\pm 0}{i \Delta Y} \to 0 \text{ along imaginary axis}
\]

Case II: \( \Delta Y \to 0 \) first (\( \Delta z \to 0 \) along imaginary axis)

\[
\Rightarrow f' \to \frac{\Delta X}{\Delta X} \to 1.
\]

With apologies to those who have seen this, I'll do two more...
EXAMPLE: SHOW THAT \( f(z) = 1/(z - c) \) IS ANALYTIC EVERYWHERE EXCEPT \( z = c \).

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left( \frac{1}{z + \Delta z - c} - \frac{1}{z - c} \right)
\]

\[
= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left( \frac{[z - c] - [z + \Delta z - c]}{[z + \Delta z - c][z - c]} \right)
\]

\[
= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left( \frac{-\Delta z}{[z + \Delta z - c][z - c]} \right)
\]

\[
= -\frac{1}{(z - c)^2}, \quad \text{INDEPENDENT OF HOW} \quad \Delta z \text{ APPROACHES ZERO.}
\]

BUT CLEARLY UNDEFINED AT \( z = c \).
Example: Show that $f(z) = |z|$ is not analytic.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \frac{|z + \Delta z| - |z|}{\Delta z}$$

Observe: Numerator $\in \mathbb{R}$

Denominator $\in \mathbb{C}$

So if $\Delta z \to 0$ along real axis, $f' \in \mathbb{R}$

If $\Delta z \to 0$ along imaginary axis, $f' \notin \mathbb{R}$

$\therefore f'$ is undefined. QED

More examples are in RH...
Suppose \( f(z) \) is analytic in some domain \( D \), where we define

\[
 f' = \lim_{\Delta z \to 0} \left[ \frac{f(z + \Delta z) - f(z)}{\Delta z} \right], \quad z \in D
\]

Letting \( f(z) = u(x, y) + iv(x, y) \), this becomes

\[
 f' = \lim_{\Delta z \to 0} \left[ \frac{U(x + \Delta x, y + \Delta y) - U(x, y) + iv(x + \Delta x, y + \Delta y) - iv(x, y)}{\Delta x + i \Delta y} \right]
\]

I. Suppose \( \Delta z \to 0 \) along real axis, i.e., \( \Delta y \equiv 0 \):

\[
 f' = \lim_{\Delta x \to 0} \left[ \frac{U(x + \Delta x, y) - U(x, y)}{\Delta x} + i \frac{V(x + \Delta x, y) - V(x, y)}{\Delta x} \right]
\]

\[
 \Rightarrow f' = \frac{du}{dx} + i \frac{dv}{dx}
\]

II. Suppose \( \Delta z \to 0 \) along imaginary axis, i.e., \( \Delta x \equiv 0 \):

\[
 f' = \lim_{\Delta y \to 0} \left[ \frac{U(x, y + \Delta y) - U(x, y)}{i \Delta y} + i \frac{V(x, y + \Delta y) - V(x, y)}{i \Delta y} \right]
\]

\[
 \Rightarrow f' = -i \frac{du}{dy} + \frac{dv}{dy}
\]

Since \( f(z) \) is supposed to be analytic, cases I and II must agree!
\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}

\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}
\end{align*}

\text{CAUCHY-RIEMANN RELATIONS}

\text{SINCE THESE FOLLOW FROM ASSUMING } f(z) \text{ ANALYTIC, THEY ARE A NECESSARY BUT NOT SUFFICIENT CONDITION FOR } f(z) \text{ TO BE ANALYTIC.}

\text{TURNS OUT THAT EXISTENCE AND CONTINUITY OF } \frac{\partial}{\partial x} u, \frac{\partial}{\partial y} u, \frac{\partial}{\partial x} v, \frac{\partial}{\partial y} v \text{ PLUS THE C-R RELATIONS, ARE SUFFICIENT TO SHOW THAT } f(z) \text{ IS ANALYTIC.}
CCK CONJECTURE: IF \( f(z) \) CAN BE EXPRESSED DIRECTLY IN TERMS OF \( z \) [WITHOUT REFERENCE TO \( x \) OR \( y \), OR ARTIFICES THAT WOULD ALLOW US TO TREAT \( x \) & \( y \) DIFFERENTLY SUCH AS ||, \( \text{Re}(\cdot) \), \( \text{Im}(\cdot) \), \( * \)] THEN \( f(z) \) IS PROBABLY ANALYTIC (OR CAN BE DEFINED SO THAT IT IS).

EXAMPLE

\[
f(z) = |x| - i|y|
\]

IN QUADRANT I, \( f(z) = x - iy = z^* \)

\[\begin{align*}
\text{II, } f(z) &= -x - iy = -z^* \\
\text{III, } f(z) &= -x + iy = -z \\
\text{IV, } f(z) &= x + iy = z
\end{align*}\]

CCK INTUITS: \( f(z) \) ANALYTIC IN QUADRANTS II & IV ONLY.

CHECK USING CAUCHY-RIEMANN!

ANALYTIC \( f(z) \) ARE THE INTERESTING KIND, AS FAR AS THE FIELD OF COMPLEX VARIABLES IS CONCERNED.

ANALYTIC = GOOD.