AIRFOIL THEORY (2D)

\( \vec{F}(\theta) \)

\( V_\infty \)

\( V_\infty \)

WHAT EQUATION & B.C.'S CAN WE USE TO DESCRIBE \( \vec{F}(\theta) \) IN THE NEIGHBORHOOD OF THE AIRFOIL? HOW CAN WE CALCULATE LIFT?

OUTLINE
- POTENTIAL FLOW
- FLOW AROUND A CYLINDER
- JOUKOWSKY AIRFOILS
- KUTTA-JOUKOWSKY THEOREM (LIFT)
Navier-Stokes:

Potential Incompressible Flow

\[ \rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \nabla p - \nabla^2 \vec{v} \]

Gravitational Potential

\[ \rho \left[ \frac{D\vec{v}}{Dt} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\rho \nabla \Psi - \nabla p - \mu \nabla^2 \vec{v} \]

Steady Inviscid: \( \mu \rightarrow 0 \)

Now consider the evolution of the vorticity \( \vec{\omega} = \nabla \times \vec{v} \).

Take the curl of *:\

\[ \frac{\partial \vec{\omega}}{\partial t} + \nabla \times [ (\vec{v} \cdot \nabla) \vec{v} ] = 0 \]

Identity: \( (\vec{v} \cdot \nabla) \vec{v} = \frac{1}{2} \nabla v^2 - \vec{v} \times \vec{\omega} \), so

\[ \frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) \]

If initially \( \vec{\omega} \equiv 0 \) then \( \vec{\omega} \equiv 0 \) forever. (Moreover, viscosity tends to reduce \( \vec{\omega} \)!) So let's assume \( \nabla \times \vec{\omega} = 0 \).

Further, let us assume incompressible flow: \( \nabla \cdot \vec{v} = 0 \), \( \rho \) constant (equivalently, the flow is subsonic, \( V_{\infty} \ll C_s \)).
EQUATION 4 is now:

\[ \rho \left[ \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla v^2 \right] = -\rho \nabla \Psi - \nabla p \]

For our airfoil, we seek a STeady flow solution, \( \frac{\partial}{\partial t} = 0 \):

\[ \nabla \left( \frac{1}{2} \rho v^2 + \rho \Psi + P \right) = 0 \]

So, the quantity \( H = \frac{1}{2} \rho v^2 + \rho \Psi + P \) is a GLOBAL constant of the flow (compare Bernoulli's Law).

This allows us to calculate pressure everywhere, given \( \vec{v}(\vec{x}) \). \( \vec{v} \) itself must satisfy

\[ \nabla \times \vec{v} = 0, \quad \nabla \cdot \vec{v} = 0. \]

GENERAL SOLUTION:

\[ \vec{v} = \nabla \phi, \quad \nabla^2 \phi = 0. \]

We call this a "POTENTIAL FLOW."
FLOW AROUND A CYLINDER

RECALL THAT $f(z)$ IS A SOLUTION TO $\nabla^2 f = 0$.

LET $f_1(z) = V_\infty (z + \frac{i}{2})$, $\vec{V}_1 = \nabla \text{Re}(f_1(z))$

UNIFORM VELOCITY OF $V_\infty \hat{x}$

SOURCE DOUBLET (2D DIPOLE)

CONTOURS OF $\text{Im}(f_1)$ ARE STREAMLINES.

ADDED TOGETHER:

UNIT CIRCLE IS A STREAMLINES.

NOW CONSIDER $f_2(z) = \frac{i \int e^{i\theta} \, d\theta}{2\pi}$, $\vec{V}_2 = \nabla \text{Re}(f_2(z))$

$\vec{V}_2 = -\frac{d}{dr} \left( \frac{\Gamma \ln r}{2\pi} \right) \phi = -\hat{\phi} \frac{\Gamma}{2\pi r}$

$\oint_{r=R} \vec{V}_2 \cdot d\ell = \bar{\Gamma} = -\frac{\Gamma}{2\pi R}$

NOTE THAT $\nabla \times \vec{V}_2 = \hat{\phi} \Gamma S(\hat{r})$.

(Zero except at origin)

LET $F(z) = f_1 + f_2$:

(Discuss how appearance changes with increasing $\Gamma$ and sailing applications!)
Now let \( F(z) = f_1 + f_2 \):

In practice, the circulation is introduced by spinning the cylinder.

There is an increasing upward "Magnus" force with increasing circulation.

- Anton Flettner patented the "Rotor Ship" in 1922.
- Cousteau's ship Alcyone - 1935.
  (His "Turbosail" is a little more complicated than Flettner's design)

→ How might we calculate the Magnus force?

→ What is the Electrostatic Analogy?
**Joukowski Transform**

(An example of conformal mapping)

\[ \xi = \xi_1 + \xi_2 \]

**\( \xi\)-plane**

\[ F(\xi) = f_1(\xi) + f_2(\xi) \]

**\( \z\)-plane**

\[ F(z) = F(\xi(z)) \]

\[ z = \xi + \frac{1}{\xi} \]

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Now have a look at the examples on the web.

**Correct \( \Gamma \):**

\[ \Gamma \text{ too small:} \]

\[ \Gamma \text{ too large:} \]

**Kutta-Joukowski Condition:**

\( \Gamma \) should be chosen to put the trailing stagnation point on the cusp of the airfoil.

- This avoids \( \vec{U} \rightarrow \infty \)
- Momentum dominates over viscosity (high Reynolds number)
- This is what happens naturally with a sharp trailing edge!

Note that the boundary conditions are ONUX

"Infinity Slip" — potential \( \nabla \Phi \cdot \hat{n} = 0 \), or

Stream func. \( \nabla \Phi \times \hat{n} = 0 \)
Q: HOW TO CALCULATE LIFT?

A: KUTTA-JOUKOWSKY THEOREM

\[
F_L = \rho V \infty \Gamma L
\]

FLUID DENSITY WIND SPEED CIRCULATION WING LENGTH

Q: IS THIS PLUSSIBLE? DOES WING SHAPE COME INTO IT?!

A: WING SHAPE DETERMINES \( \Gamma \) VIA KJ CONDITION.

THE TRADITIONAL DERIVATION IS A BIT TROUBLESOME. USES INTEGRATION OF PRESSURE FORCE OR AIRFOIL IN COMPLEX PLANE, REQUIRES RESIDUE THEOREM, AND CAREFUL DISSECTION OF A LAURENT SERIES.

(WHEN YOU HAVE A HAMMER IN YOUR HAND, LOTS OF THINGS START TO LOOK LIKE NAILS!)

SIMPLER DERIVATION — FIND THE RATE AT WHICH THE AIRFOIL IMPARTS MOMENTUM TO THE FLUID.

IN OTHER WORDS, IT'S A SCATTERING PROBLEM...
Q: If $\vec{v} \rightarrow V_\infty \hat{x}$ far from airfoil for $x \rightarrow \pm \infty$, how can it be that momentum was imparted to the air?
A: The same way $\oint \vec{a} \cdot d\vec{a} = Q_{enc}/\varepsilon_0$ as $r \rightarrow \infty$.

$\text{y-momentum per volume: } \rho V_y$

$\text{y-momentum per time, per area: } V_\infty \rho V_y$

$\text{Incoming y-momentum per time: } \sum_A dA \rho V_\infty V_y(x_i) = \int L \rho V_\infty \int_{-\infty}^{\infty} V_y(x_i,y) dy$

$\text{Outgoing y-momentum per time: } L \rho V_\infty \int_{-\infty}^{\infty} V_y(x_2,y) dy$

$-\Delta \rho \text{ per time = Lift} = L \rho V_\infty \int_{-\infty}^{\infty} (V_{y1} - V_{y2}) dy$

$= L \rho V_\infty \Gamma \quad \Box_e.$
How to model a wing

- Pick a Joukowsky profile of desired shape
- Set angle of attack by rotating the $f$-term in the cylinder flow:
  \[ f_1 = V_\infty \left( z e^{i\alpha} + \frac{1}{\bar{z} e^{i\alpha}} \right) \]
- Set $\Gamma$ using Kutta-Joukowsky condition
- Calculate lift using K-J theorem, $F_L = \frac{1}{2} \rho V_\infty \Gamma$

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Q: Is $\Gamma$ invariant under Joukowsky transform?

A: $\Gamma$ is invariant under conformal mapping:

Let $W(z) = \frac{dF(z)}{dt}$, $\tilde{W}(\xi) = W(z) \frac{dz}{d\xi} = \frac{dF(\xi)}{d\xi}$

$\Gamma = \text{Re} \left[ \oint W(z) dz \right]$ (might want to show this!)

$\oint W(z) dz = \oint \tilde{W}(\xi) \frac{dz}{d\xi} d\xi = \oint \tilde{W}(\xi) d\xi$

$\therefore \quad \Gamma = \tilde{\Gamma}$ (QED)