So far in Ch. 24

§ 24.1 Analytic \( f(z) = u + iv \)
§ 24.2 Cauchy-Riemann Relations
§ 24.5 Branch Points & Branch Cuts for Multi-valued \( f(z) \)
§ 24.7 \( u \) & \( v \) as Soln's to Laplace's Equation
Stream Functions

Today
§ 24.3 Complex Power Series
§ 24.6 Singularities & Zeros

Evaluations?

Today's lecture provides the background we need for integration in the complex plane §§ 24.8 ff.
§24.3 - CONVERGENCE OF COMPLEX POWER SERIES

Suppose we have

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C} \]

\[ = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \]

This series will certainly converge if the following converges:

\[ \sum_{n=0}^{\infty} |a_n| r^n \]

The convergence of this real series can be evaluated using the Cauchy root test: \((\S 4.5.1)\)

The series converges for \(|z| < R\),

where \( R = \lim_{n \to \infty} |a_n|^{1/n} \)

\( R \) is called the radius of convergence.

(For \(|z| = R\), you just have to look carefully)

Alternatively,

\[ R = \lim_{n \to \infty} \left| \frac{|a_n|}{|a_{n+1}|} \right| \] (Ratio Test)
Example — Find the radius of convergence:

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

\[ |a_n| = \frac{1}{n!} \]

\[ R = \lim_{n \to \infty} |a_n|^{-1/n} = \lim_{n \to \infty} \left| \frac{n!}{1/n} \right| \]

\[ \text{Much easier with the ratio test} \]

For large \( n \), \( n! \to \sqrt{2\pi n} \frac{n^ne^{-n}}{\sqrt{n!}} \)

So

\[ R = \lim_{n \to \infty} \left[ \sqrt{2\pi n} \frac{n^ne^{-n}}{\sqrt{n!}} \right]^{1/n} \]

\[ = \lim_{n \to \infty} \sqrt{2\pi} \frac{n^{1/2}n^{-1}e^{-1}}{\sqrt{n!}} \]

\[ \to \infty \]

So the series converges for all \( z \).

More examples in RH2:

\[ \sum_{n=0}^{\infty} n! z^n \rightarrow R = 0 \]

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \rightarrow R = 1 \]
§ 24.6 SINGULARITIES & ZEROS OF $f(z)$

**Singularity:** Any point where $f(z)$ is not analytic.

If the singularity is confined to a point, it is called an isolated singularity.

Q: For $\ln z$, the origin is a singularity. Is it an isolated singularity?

A: No. It is a branch point, and $\ln z$ is also not analytic along the branch cut.

Branch points are not isolated singularities.

A pole of order $n$ is an isolated singularity of the form

$$f(z) = \frac{g(z)}{(z-z_0)^n}$$

More precisely, we mean that

$$\lim_{z \to z_0} \left[ (z-z_0)^n f(z) \right] = g(z_0), \quad \text{where}$$

- $g(z_0) \in \mathbb{C}$ is finite, $\neq 0$
- A pole of order $n=1$ is a simple pole
- If $g(z_0) = 0$, we have a pole of order $< n$ or $f(z)$ is analytic at $z_0$.
- If $g(z_0) \to \infty$, it is a pole of order $> n$.
- If the limit is $\infty$ regardless of $n$, the singularity is essential.
Final type of singularity:

\[ f(z) \to 0 \text{ or } \infty \text{ as } z \to z_0 \]

But \( \lim_{z \to z_0} f(z) \) exists and is well-defined,

then the singularity is "removable".
EXAMPLE
IDENTIFY & CLASSIFY THE SINGULARITIES:

\[ f(z) = \frac{1}{1 - z} - \frac{1}{1 + z} = f(z) \]

\[ f'(z) = \tanh(z) = \frac{\sinh z}{\cosh z} = \frac{(e^z - e^{-z})/2}{(e^z + e^{-z})/2} \]

\[ f'(z) = \frac{\sin z}{z} \]

(1) POLE AT \( z = 1 \):
\[ \left[ f(z), (z-1) \right] = \left[ -1, -\frac{z-1}{1+z} \right] \]

\[ z = 1 \]

\[ \Rightarrow \text{SIMPLE POLE AT } z = 1 \]

(SIMILARLY AT \( z = -1 \))

(2) \[ \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \]

SINGULARITIES SATISFY \( e^z = -e^{-z} \)

\[ = e^{i(2m+1)\pi} e^{-z}, \quad m \in \mathbb{Z} \]

\[ \Rightarrow z = i\pi \left( m + \frac{1}{2} \right) \]

TRY \( n = 1 \): \[ \lim_{z \to i\pi \left( m + \frac{1}{2} \right)} \left[ \frac{z - i\pi \left( m + \frac{1}{2} \right)}{\sinh z} \right] = ? \]

SIMPLE POLE

USE L'HOSPITAL'S RULE
\[ = \lim_{z \to i\pi \left( m + \frac{1}{2} \right)} \left[ \frac{z - i\pi \left( m + \frac{1}{2} \right)}{\sinh z} \right] = 1 \]
\[ f(z) = \sin \frac{z}{z} \]

\[ f(z) = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \]

\[ = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \]

**THE SINGULARITY IS REMOVABLE.**
ZERES OF $f(z)$ ARE CALLED LIKE POLES.

If $f(z) = (z - z_0)^n g(z)$, $g(z)$ FINITE AT $z = z_0$,

THEN $f(z)$ HAS A ZERO OF ORDER $n$ AT $z_0$. 