You may use one 8.5 × 11 sheet of notes during the final exam.
No other reference materials or notes will be allowed.
Fourier transform pairs:

\[
\begin{array}{c|c}
\text{f(x)} & \tilde{f}(k) \\
1 & \sqrt{2\pi}\delta(k) \\
e^{iax} & \sqrt{2\pi}\delta(k - a) \\
ed^{-a|x|} & \sqrt{2\pi} \frac{a}{a^2 + k^2} \\
e^{-a^2x^2}, \ a > 0 & \frac{1}{a\sqrt{2}} e^{-k^2/4a^2} \\
H(x) & \frac{1}{ik\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}}\delta(k)
\end{array}
\]

The first few \(Y_l^m\):

\[
\begin{align*}
Y_0^0 &= \sqrt{\frac{1}{4\pi}} \\
Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos \theta \\
Y_{1}^{\pm1} &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \\
Y_2^0 &= \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1) \\
Y_2^1 &= \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \\
Y_2^{\pm2} &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}
\end{align*}
\]

Differential operators in spherical symmetry:

\[
\begin{align*}
\nabla f(r) &= \frac{df}{dr} \hat{r} \\
\nabla \cdot [\hat{r} f(r)] &= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r)\right] \\
\nabla^2 f(r) &= \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{df}{dr}\right]
\end{align*}
\]
1. *(18 points)* The fundamental equation of mechanics is Newton’s second law,

\[ m \frac{d^2 x}{dt^2} = F(t). \]  

(1)

We will assume that the net force \( F \) is some known function of time.

(a) Find the Green’s function for equation (1), using the initial conditions \( x = 0 \) and \( \dot{x} = 0 \) at \( t = 0 \).

(b) Use the Green’s function to find the trajectory \( x(t) \) on \( t > 0 \) for the applied force

\[ F(t) = \begin{cases} 
0, & t < \tau; \\
F_0 \left( \frac{t}{\tau} \right)^2, & t > \tau.
\end{cases} \]
2. (20 points) Solve by Fourier transform:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,
\]

subject to

\[
u(0, y) = \cos(y), \quad \frac{\partial u}{\partial x} \bigg|_{x=0} = 0.
\]
3. (20 points) The output voltage $V(t)$ of a filtering circuit obeys the equation

$$LRC(\dot{\Phi} - \dot{V}) + R(\Phi - V) - LV = 0,$$

where $\Phi(t)$ is some input signal and $L, R, C$ are constants.

(a) Find the complex frequency response $T(\omega)$, defined by $\tilde{V} = T\tilde{\Phi}$.

(b) Find all zeroes and poles of $T(\omega)$, taking $\omega \in \mathbb{C}$.

(c) Sketch the amplitude response, $|T(\omega)|$, for $\omega \in \mathbb{R}^+$. Note the limits $\omega \to 0$ and $\omega \to \infty$.

(d) From your solution, identify the Fourier transform of the Green’s function.
4. (20 points) let \( f(z) = \frac{1}{1 + z^2} \).

(a) Where is \( f(z) \) analytic?

(b) It is easy to show using trigonometric substitution that

\[
\int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}.
\]

Use this fact and integrate \( f(z) \) on the illustrated contour to show that

\[
\int_0^\infty \frac{1 + r^2}{1 + r^4} \, dr = \frac{\pi}{2\sqrt{2}} \quad \text{and} \quad \int_0^\infty \frac{1 - r^2}{1 + r^4} \, dr = 0.
\]
5. (15 points) The diffusion of an instantaneous point deposit of heat at time \( t = t_0 \) and position \( r = r_0 \) in an infinite medium corresponds to the Green's function

\[
G(r, t, r_0, t_0) = H(t - t_0) \left[ 2a \sqrt{\pi(t - t_0)} \right]^{-3} \exp \left\{ -(r - r_0)^2 / [4a^2(t - t_0)] \right\}.
\]

From this, develop two modified Green's functions obeying homogeneous Dirichlet and Neumann boundary conditions on \( z > 0 \),

(a) \( G|_{z=0} = 0 \).

(b) \( \frac{\partial G}{\partial z} \bigg|_{z=0} = 0 \).
6. (20 points) A Hermitian partial differential operator $\mathcal{D}$ has eigenvalues $\lambda_n$ and properly normalized eigenfunctions $u_n(r)$.

(a) Expand an arbitrary function $f(r)$ in terms of the eigenfunctions. Use Fourier’s trick to solve for the coefficients, $c_n$.

(b) Express $\delta(r - r_0)$ in terms of the eigenfunctions.

(c) Show that

$$\sum_n \frac{1}{\lambda_n} u^*_n(r_0) u_n(r)$$

is the Green’s function of $\mathcal{D}$. 
7. Solve the scalar wave equation, $u_{xx} = u_{tt}/c^2$, by the method of separation of variables. Divide the general solution into two pieces, one that depends on $x - ct$, and one that depends on $x + ct$. Manipulate further to obtain $u(x, t) = f(x - ct) + g(x + ct)$, where $f$ and $g$ are any functions expressible as inverse Fourier transforms.