

## ASIDE ON QUADRATURE

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NOTE THAT THE 1D QUADRATURE PROBLEM,

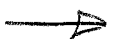
$$y = \int_a^x F(x') dx',$$

IS EQUIVALENT TO A 1<sup>ST</sup> ORDER INITIAL VALUE PROBLEM:

$$\frac{dy}{dx} = F(x), \quad y(a) = 0.$$

WE ALSO KNOW THAT AN M-DIMENSIONAL QUADRATURE PROBLEM IS, IN PRINCIPLE, JUST M NESTED 1D INTEGRALS.

- ODE SOLVERS CAN BE USED TO SOLVE INTEGRALS NUMERICALLY.
  - SINCE ODE SOLVERS NORMALLY RECORD THE TRAJECTORY  $y(x)$ , THEY ARE CONVENIENT FOR PERFORMING INDEFINITE INTEGRALS.
  - ODE SOLVERS OFTEN HAVE VARIABLE STEPSIZE.
  - FOR MOST PURPOSES, IF YOU HAVE A GOOD ODE SOLVER, JUST USE IT FOR QUADRATURE TOO!
- ★ MOST ODE SOLVERS, HOWEVER, INSIST ON EVALUATING  $f$  AT THE ENDPOINTS. SO, HANG ON TO TRES HERMANOS AND/OR GAUSSIAN QUADRATURE FOR THOSE PESKY IMPROPER INTEGRALS!



(QUADRATURE, CONT.)

(2)

HOW DO OUR ODE SOLVERS COMPARE WITH THE QUADRATURE ROUTINES WE LEARNED ABOUT EARLIER?

EULER

WANT  $\int_a^b f(t) dt$ . LET  $x(t) = \int_a^t f(t') dt'$ .

THE CORRESPONDING ODE INITIAL VALUE PROBLEM IS:

$$\frac{dx}{dt} = f(t), \quad x(a) = 0$$

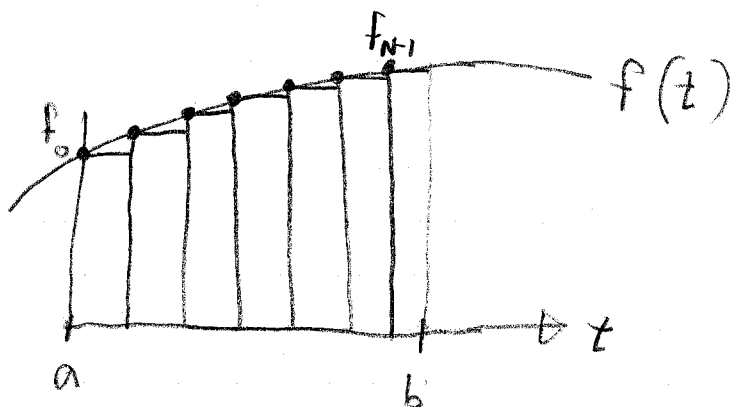
LET'S USE  $N$  POINTS,  $h = (b-a)/N$ .

$$t_n = a + nh, \quad x_n = x(t_n), \quad x_0 = 0$$

EULER:  $x_{n+1} = x_n + h \underbrace{f(t_n)}_{f_n}$

THE DESIRED RESULT IS  $x_N = x(b) = 0 + h(f_0 + f_1 + \dots + f_{N-1})$ .

$$\int_a^b f dt \approx h \sum_{n=0}^{N-1} f_n$$



THIS IS NOT EVEN FIRST ORDER!

(QUADRATURE, CONT.)

RK2

$$X_{n+1} = X_n + h f\left(X_n + \frac{h}{2} \dot{X}_n, t_n + \frac{h}{2}\right)$$

BUT FOR THE QUADRATURE APPLICATION,  $f(x, t) \rightarrow f(t)$ .

$$X_{n+1} = X_n + h f\left(t_n + \frac{h}{2}\right)$$

SO

$$\int_a^b f dt \approx X_N = 0 + h\left(f\left(a + \frac{1}{2}\right) + f\left(a + \frac{3}{2}\right) + \dots\right)$$
$$= \sum_{n=0}^{N-1} f\left(a + n - \frac{1}{2}\right)$$

THAT'S THE EXTENDED MIDPOINT RULE!\*

- THE OTHER RK2 METHOD TURNS OUT TO BE THE EXTENDED TRAPEZOID RULE.

THE GENERAL PATTERN, THEN, IS THAT A QUADRATURE THAT REPRESENTS THE INTEGRAND TO FIRST ORDER (MIDPOINT, TRAPEZOID) ARE EQUIVALENT TO SECOND ORDER PDE SOLVERS (RK2), BECAUSE THE INFINITE INTEGRAL IS GOOD TO SECOND ORDER.

(RECALL THAT MIDPOINT HAD A LEADING ERROR TERM OF  $\mathcal{O}(h^3)$ . IT IS MORE CONSISTENT TO CALL IT A 2<sup>nd</sup> ORDER ROUTINE.)

SO SINCE TRES HERMINOS HAS ERROR  $\mathcal{O}(h^5)$ , IT IS A 4<sup>th</sup> ORDER ROUTINE, COMPARABLE TO RK4.

\*Q: IS THIS RK2 AN "OPEN" METHOD? A: YES, AS LONG AS  $f(x, t) = f(t)$ .

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