**STEP DOUBLING**

**GOALS:**

1. **DEVELOP MACHINERY TO SUPPORT VARIABLE STEPSIZE, h.**
2. ** PROVIDE AN ERROR ESTIMATE ON THE FLY.**
3. **PARLAY AN \( m^{th} \) - ORDER ROUTINE INTO \( m+1 \) ST -ORDER.**

I ** WILL DO IT IN A CHEESY WAY. MY APPROACH WILL BE EFFECTIVE, AND HOPEFULLY CLEAR, BUT FAR FROM OPTIMAL.**

**PLAN:**

- **STEP DOUBLING FOR \( m^{th} \) - ORDER RK**
  - **EXAMPLE: DOUBLED EULER** (RK12)
  - **A VARIABLE-STEP SOLVER, WITH EXAMPLES (I HOPE!)**
GENERALIZED STEP DOUBLING

GIVEN AN m-TH ORDER RK,

\[ X_1 = X_0 + R_m(f, X_0, t_0, h) + C h^m + O(h^{m+1}) \]

USE TWO HALF-STEP TO FORM ANOTHER ESTIMATE OF X_1:

\[ X_{1/2} = X_0 + R_m(f, X_0, t_0, h/2) + C \left( \frac{h}{2} \right)^m + O(h^{m+1}) \]

\[ X_1 = X_{1/2} + R_m(f, X_{1/2}, t_{1/2}, h/2) + C \left( \frac{h}{2} \right)^m + O(h^{m+1}) \]

\[ = [X_{1/2} + C \left( \frac{h}{2} \right)^m] + R_m(f, X_{1/2}, t_{1/2}, h/2) + C \left( \frac{h}{2} \right)^m + O(h^{m+1}) \]

\[ = X_{1/2} + R_m(f, X_{1/2}, t_{1/2}, h/2) + 2C \left( \frac{h}{2} \right)^m + O(h^{m+1}) \]

\[ X_{1**}, \text{SECOND ESTIMATE OF } X_1 \]

NOW, LOOK AT THE DIFF. BETWEEN OUR 2 ESTIMATES OF X_1:

\[ X_{1**} - X_1 = C h^m \left[ 1 - \frac{1}{2^m} \right] = C h^m \left( \frac{2^m-1}{2^m} \right) \]

FROM THIS WE INFERENCE TRUNCATION ERROR OF OUR SECOND ESTIMATE:

\[ \varepsilon = \frac{1}{2^m} C h^m = \frac{(X_{1**} - X_1)}{(2^m-1)} \]

... AND USE THIS TO ELIMINATE m+1ST ORDER TRUNCATION ERROR:

\[ X_1 = X_{1**} + \varepsilon + O(h^{m+2}) = X_{1**} \left( \frac{2^m}{2^m-1} \right) - X_1 \left( \frac{1}{2^m-1} \right) \]

* BOTH THESE TERMS ALSO CONTRIBUTE TO THE O(h^{m+2}).
**Simple Example: Doubled Euler (RK12)**

\[
X_1^* = X_0 + h f(x_0, t_0), \quad X_1 = X_1^* + \frac{h^2}{2} \mathcal{O}(h^2)
\]

\[
X_{1/2}^* = X_0 + \frac{h}{2} f(x_0, t_0)
\]

\[
X_{1/2} = X_{1/2}^* + \frac{h}{2} f(x_{1/2}^*, t_{1/2})
\]

\[
X_1 = X_{1/2}^* \left( \frac{2}{2-1} \right) - X_{1/2} \left( \frac{1}{2-1} \right) + \mathcal{O}(h^3)
\]

\[
= 2 \left( X_{1/2}^* + \frac{h}{2} f(x_{1/2}^*, t_{1/2}) \right) - \left( X_0 + h f(x_0, t_0) \right)
\]

\[
X_1 = X_0 + h f(x_{1/2}^*, t_{1/2}) + \mathcal{O}(h^3)
\]

This is just RK2!

- Double as needed until \( |e| < \text{specified tolerance} \).
- Note that the error estimate is very conservative:

\[
E = \frac{X_{1/2}^* - X_{1/2}}{2} = \mathcal{O}(h^2)
\]

- Professional grade solutions (RK45) arrange the RKm so that \( X_1^* \) uses \( f \) evaluated in \( m \) of the same locations as \( X_{1/2}^* \). So there are no unnecessary function calls. Furthermore, successive doublings inherit previous results, so \( f \) is never unnecessarily evaluated at the same \( x_i \) twice.