

ASIDE

CONSIDER A REAL $N \times N$ MATRIX M .

SUPPOSE $M\vec{\eta} = \lambda\vec{\eta}$, $\lambda = a+ib$; $a, b \in \mathbb{R}$

REPRESENT $\vec{\eta} = \vec{\alpha} + i\vec{\beta}$, $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^N$.

★ NOTE THAT IF $b \neq 0$, $\vec{\beta} \neq 0$. (λ COMPLEX $\Rightarrow \vec{\eta}$ COMPLEX).

$$\begin{aligned} \text{NOW, } M\vec{\eta} &= M(\vec{\alpha} + i\vec{\beta}) = (\vec{\alpha} + i\vec{\beta})(a + ib) \\ &= (a + ib)\vec{\alpha} + (ia - b)\vec{\beta}. \end{aligned}$$

CONSEQUENTLY,

$$\text{Re}[M\vec{\eta}] = M\vec{\alpha} = a\vec{\alpha} - b\vec{\beta},$$

$$\text{Im}[M\vec{\eta}] = M\vec{\beta} = b\vec{\alpha} + a\vec{\beta}.$$

$$\begin{aligned} \Rightarrow M\vec{\eta}^* &= M(\vec{\alpha} - i\vec{\beta}) = (a\vec{\alpha} - b\vec{\beta}) - i(b\vec{\alpha} + a\vec{\beta}) \\ &= (a - ib)\vec{\alpha} + (-b - ia)\vec{\beta} \\ &= (a - ib)(\vec{\alpha} - i\vec{\beta}) \end{aligned}$$

$$\therefore \boxed{M\vec{\eta}^* = \lambda^*\vec{\eta}^*}$$

(CONT.)

SINCE THE EIGENVALUES ARE DISTINCT (IF $b \neq 0$),
THE TWO EIGENVECTORS ARE ORTHOGONAL:

$$\langle \vec{\alpha} + i\vec{\beta} \mid \vec{\alpha} - i\vec{\beta} \rangle = 0$$

IN TERMS OF THE CONVENTIONAL "DOT" PRODUCT,

$$(\vec{\alpha} - i\vec{\beta}) \cdot (\vec{\alpha} - i\vec{\beta}) = 0$$

$$\alpha^2 - \beta^2 - 2i(\vec{\alpha} \cdot \vec{\beta}) = 0$$

$$\boxed{\alpha^2 = \beta^2, \quad \vec{\alpha} \cdot \vec{\beta} = 0.}$$

FINALLY, CONSIDER \vec{r} , A REAL VECTOR THAT LIVES
IN THE SUBSPACE OF $\vec{\eta}$ AND $\vec{\eta}^*$:

$$\begin{aligned} \vec{r} &= C(\vec{\alpha} + i\vec{\beta}) + D(\vec{\alpha} - i\vec{\beta}) \in \mathbb{R}^N \\ &= (C+D)\vec{\alpha} + i(C-D)\vec{\beta}. \end{aligned}$$

REQUIRE $C+D \in \mathbb{R}$, $C-D \in \mathbb{I}$

$$\Rightarrow \text{Im}(D) = -\text{Im}(C)$$

$$\& \text{Re}(C) = \text{Re}(D)$$

$$\therefore \underline{\underline{D = C^*}}$$

$$\begin{aligned} \underline{\underline{\text{SO}}} \quad \vec{r} &= (p+iq)\vec{\eta} + (p-iq)\vec{\eta}^* \\ &= 2p\vec{\alpha} - 2q\vec{\beta}. \end{aligned}$$

SYSTEMS OF COUPLED ODE'S AND CHAOS

- ODE = VECTOR FIELD IN PHASE SPACE
 - FIXED POINTS (NULLS)
 - JACOBIAN MATRIX, CLASSIFICATION OF NULLS
- THESE TOOLS WILL COME IN HANDY, AND LEAD NATURALLY INTO A BRIEF DISCUSSION OF CHAOS.

THE ODE

$$\frac{d\vec{x}}{dt} = \vec{g}(\vec{x}, t), \quad \vec{x} \in \mathbb{R}^{N-1}$$

CAN BE MADE TIME-INDEPENDENT BY DEFINING

$$\vec{r} = (\vec{x}, t), \quad \vec{r} \in \mathbb{R}^N$$

SO THE EQUATION IS NOW

$$\frac{d\vec{r}}{dt} = \frac{d}{dt} \begin{bmatrix} \vec{x} \\ t \end{bmatrix} = \begin{bmatrix} \vec{g}(\vec{x}, t) \\ 1 \end{bmatrix} \equiv \vec{f}(\vec{r})$$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{f}(\vec{r})}$$

THE SYSTEM OF COUPLED ODE'S CAN NOW BE VIEWED AS A VECTOR FIELD $\vec{f}(\vec{r})$ IN N DIMENSIONS.

SOLUTIONS OF $\boxed{\vec{f}(\vec{r}_F) = 0}$ ARE FIXED POINTS OR NULLS ($\dot{\vec{r}} = 0$).

THE JACOBIAN MATRIX,

$$\frac{\partial \dot{r}_j}{\partial r_k} = J_{jk},$$

WHEN EVALUATED AT A NULL \vec{r}_F , GIVES A RATHER COMPLETE DESCRIPTION OF THE BEHAVIOR OF $\vec{f} = \dot{\vec{r}}$ IN THE VICINITY OF THE NULL:

LET $\vec{r} = \vec{r}_F + \vec{\eta}$, $\vec{\eta}$ SMALL.

$$f_j(\vec{r}) = \sum_k \frac{\partial \dot{r}_j}{\partial r_k} \eta_k, \quad \text{I.E. } \vec{f}(\vec{r}) = J \vec{\eta}.$$

$$\Rightarrow \dot{\vec{r}} = \boxed{\dot{\vec{\eta}} = J \vec{\eta}} \quad \star \left. \begin{array}{l} \text{N COUPLED, LINEAR} \\ \text{ODE'S GOVERN } \vec{r} \approx \vec{r}_F. \end{array} \right\}$$

NOW CONSIDER THE EIGENVALUE EQUATION FOR J:

$$J \vec{\eta}_\alpha = \lambda_\alpha \vec{\eta}_\alpha$$

ANY STATE $\vec{\eta}$ CAN BE WRITTEN AS A SUM OF EIGENSTATES,

$$\vec{\eta}(t) = \sum_\alpha c_\alpha(t) \vec{\eta}_\alpha$$

EXPRESS BOTH SIDES OF THE EQUATION OF MOTION (\star) IN TERMS OF THE EIGENVECTORS:

$$\sum_\alpha \dot{c}_\alpha \vec{\eta}_\alpha = \sum_\alpha c_\alpha \lambda_\alpha \vec{\eta}_\alpha \Rightarrow \dot{c}_\alpha = \lambda_\alpha c_\alpha$$

$$\Rightarrow \boxed{c_\alpha(t) = c_\alpha(0) e^{\lambda_\alpha t}} \quad \star \star$$

INTERPRETATION:

- EACH EIGENMODE EVOLVES SEPARATELY.
- THE MODE GROWS ($\lambda > 0$) OR DECAYS ($\lambda < 0$) WITH TIME CONSTANT $|\lambda^{-1}|$.

EIGENVALUES ARE ROOTS OF AN N^{TH} DEGREE CHARACTERISTIC POLYNOMIAL. IT IS THEREFORE POSSIBLE TO HAVE A COMPLEX EIGENVALUE, IN WHICH CASE ANOTHER EIGENVALUE WILL BE ITS COMPLEX CONJUGATE:

$$\lambda_1 = a + ib \quad (a, b \in \mathbb{R}, b \neq 0)$$

$$\lambda_2 = a - ib$$

THE EIGENVECTORS AND AMPLITUDES ARE ALSO CONJ. PAIRS:

$$c_1 = p + iq, \quad c_2 = p - iq$$

$$\vec{\eta}_1 = \vec{\alpha} + i\vec{\beta}, \quad \vec{\eta}_2 = \vec{\alpha} - i\vec{\beta} \quad (\vec{\alpha}, \vec{\beta} \in \mathbb{R}^N)$$

SINCE THEY ARE EIGENVECTORS, $\vec{\eta}_1 \cdot \vec{\eta}_2 = 0 \Rightarrow \vec{\alpha} \cdot \vec{\beta} = 0$.

WOLOG, LET $|\vec{\alpha}| = |\vec{\beta}| = 1$.

$$\text{LET } X \equiv \vec{\alpha} \cdot \vec{\eta} = c_1 + c_2, \quad Y \equiv \vec{\beta} \cdot \vec{\eta} = i(c_1 - c_2)$$

THE SOLUTION FOR X & Y IS OBTAINED USING $\star\star$:

$$X = (p_0 + iq_0) e^{(a+ib)t} + (p_0 - iq_0) e^{(a-ib)t}$$

$$\Rightarrow \begin{cases} X = e^{at} [2p_0 \cos bt - 2q_0 \sin bt] \\ Y = e^{at} [2q_0 \cos bt - 2p_0 \sin bt] \end{cases}$$

HARMONIC MOTION, WITH EXPONENTIAL GROWTH OR DECAY, IN XY-PLANE!

CHAOS (BY EXAMPLE)

(1)

THE LORENZ EQUATIONS

$$\dot{x} = \sigma(y-x)$$

$$\dot{y} = \rho x - y - xz$$

$$\dot{z} = xy - \beta z$$

GOOD REFERENCE:

[www-atm.physics.ox.ac.uk
/user/read/chaos/chaos.html](http://www-atm.physics.ox.ac.uk/user/read/chaos/chaos.html)

LORENZ (1963 J. ATM. SCI. ~~19 329-341~~ ^{20: 130-141}) SIMPLIFIED
MODEL OF 2D CONVECTION.

x — INTENSITY OF CONVECTIVE MOTION

y — TEMP. DIFF. BETWEEN ASCENDING & DESCENDING ~~CONVECTIVE~~ FLOWS

z — DISTORTION OF T PROFILE FROM LINEAR.

β — $4(1+\alpha)^{-1}$, α = ASPECT RATIO OF CELL

ALL > 0 { σ — PRANDTL NUMBER (ν/k) VISCOSITY/CONDUCTIVITY

ρ — (R_a/R_c) RAYLEIGH #
CRITICAL RAYLEIGH #

$$\text{JACOBIAN: } J(\vec{x}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

THE WIKIPEDIA ARTICLES ON CHAOS, THE LYAPUNOV EXPONENT, AND THE LORENZ ATTRACTOR ARE VERY HELPFUL.

FIXED POINTS:

$$\dot{x} = \sigma(y-x) = 0 \rightarrow y=x$$

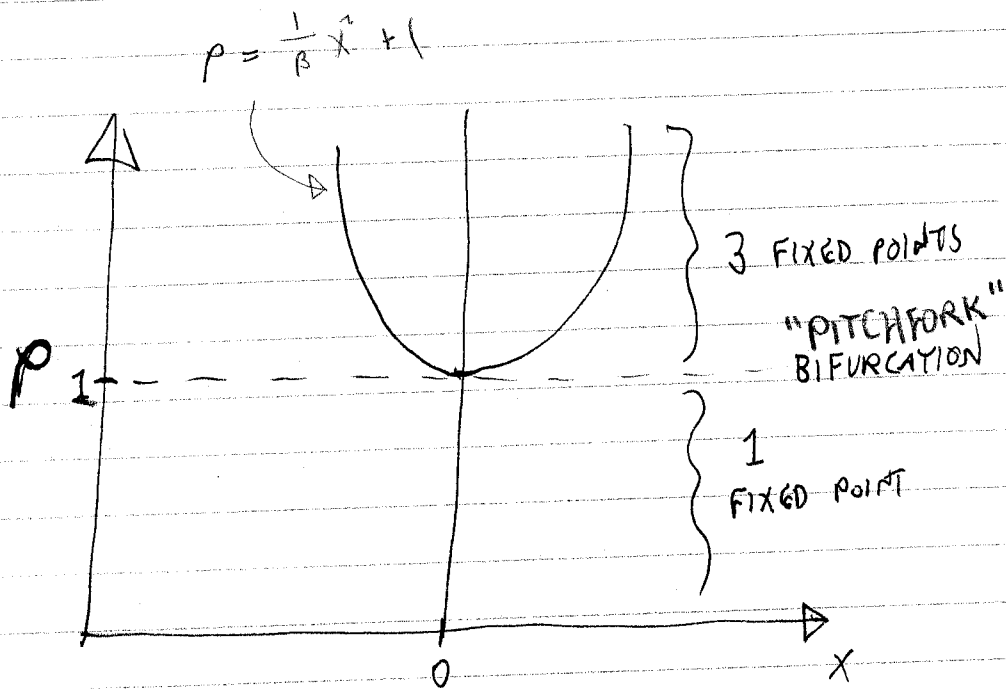
$$\dot{y} = \rho x - y - xz = 0 \rightarrow (\rho - 1 - z)x = 0$$

$$\dot{z} = xy - \beta z = 0 \rightarrow \beta z = x^2$$

ROOTS: $x=y=z=0$ ORIGIN

$$x=y = \pm \sqrt{\beta z}, \quad z = \rho - 1$$

$z = \rho - 1, \quad x=y = \pm \sqrt{\beta(\rho - 1)}$ "C⁺, C⁻"



FIXED POINT AT ORIGIN:

$$J(\vec{0}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

DECOUPLES: $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ \rho & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\frac{d}{dt} z = -\beta z$$

TRICK FOR SOLVING 2x2:

$$M \vec{x} = \lambda \vec{x}$$

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

WHERE $\tau = \text{Tr } M$
 $\Delta = \text{Det } M$

IN OUR CASE, $\tau = -(\sigma + 1)$
 $\Delta = \sigma(1 - \rho)$

$$\lambda_{\pm} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - \rho)}}{2}$$

Now, ~~$\beta > 0$~~ ~~$\beta < 0$~~ AND $\lambda_- < 0$.
 $\lambda_- < 0$ FOR ALL ρ .
 $\lambda_+ > 0$ FOR $\rho > 1$ (SADDLE)
 $\lambda_+ < 0$ FOR $\rho < 1$ (SINK)

- (4)
- IT TURNS OUT THAT FOR $\rho < 1$, ALL TRAJECTORIES FALL INTO THE ORIGIN AS $t \rightarrow \infty$.

SO ALL THE INTERESTING PHYSICS IS FOR $\rho > 1$.

- THE BEHAVIOR OF THE SYSTEM CHANGES MARKEDLY WITH ρ . GOES BACK AND FORTH AMONG A VARIETY OF BEHAVIORS
 - SOMETIMES PERIODIC ORBITS
 - PERIOD DOUBLING
 - CHAOS

- WE WILL LOOK AT JUST ONE CLASSIC CHOICE OF PARAMETERS:

$$\beta = 8/3$$

$$\sigma = 10$$

$$\rho = 28$$

- EXAMPLE USING GNU OCTAVE \rightarrow SENSITIVITY TO INITIAL COND.

LIKE MATLAB

VECTOR/MATRIX ORIENTED

INTERPRETED

POWERFUL NUMERICAL PACKAGES

LAPACK (LINEAR ALGEBRA)

LSODE — INCLUDES STIFF SOLVER

ETC.

BUILT-IN PLOTTING (GNU PLOT)

CHAOTIC SYSTEM WORKING DEFINITION:

- ① DETERMINISTIC (IN PRINCIPLE; I.E. NOT STOCHASTIC)
- ② NO STABLE FIXED POINT OR RUNAWAY CONDITION.
- ③ SOLUTIONS NOT PERIODIC. NO LIMIT CYCLES.
- ④ SENSITIVITY TO INITIAL CONDITIONS.

⚡ UNIVERSALLY ACCEPTED DEFINITION OF CHAOS.

DISCUSSION

- GIVEN THE ABOVE DEFINITION, IS THE GRAVITATIONAL 3-BODY SYSTEM CHAOTIC?

- HOW BIG DOES THE PHASE SPACE HAVE TO BE TO ACCOMMODATE CHAOTIC BEHAVIOR? \mathbb{R}^2 ? \mathbb{R}^3 ?

- LYAPUNOV EXPONENT: CONSIDER 2 NEARBY TRAJECTORIES:

$$|\delta \vec{r}| = |\vec{r}_2(t) - \vec{r}_1(t)| \approx |\delta \vec{r}(0)| e^{\lambda t}$$

HOW COULD YOU CALCULATE A "LOCALIZED" LYAPUNOV EXPONENT FROM THE JACOBIAN?