Systems of Coupled ODE's and Chaos

- ODE = Vector Field in Phase Space
- Fixed Points (Nulls)
- Jacobian Matrix, Classification of Nulls

These tools will come in handy, and lead naturally into a brief discussion of chaos.

The ODE

\[
\frac{d\dot{x}}{dt} = \ddot{g}(\dot{x}, t), \quad \dot{x} \in \mathbb{R}^{n-1}
\]

Can be made time-independent by defining

\[
\bar{r} = (\dot{x}, t), \quad \bar{r} \in \mathbb{R}^n
\]

So the equation is now

\[
\frac{d\bar{r}}{dt} = \frac{d}{dt} \left[ \begin{array}{c} \dot{x} \\ t \end{array} \right] = \left[ \begin{array}{c} \ddot{g}(\bar{r}, t) \\ 1 \end{array} \right] = \bar{f}(\bar{r})
\]

\[
\frac{d\bar{r}}{dt} = \bar{f}(\bar{r})
\]

The system of coupled ODE's can now be viewed as a vector field \( \bar{f}(\bar{r}) \) in \( n \) dimensions.

Solutions of \( \bar{f}(\bar{r}) = 0 \) are fixed points or nulls \( (\bar{r} = 0) \).
The Jacobian matrix,

\[ \frac{\partial \mathbf{\dot{r}}}{\partial \mathbf{r}} = \mathbf{J} \]

when evaluated at a null \( \mathbf{r}_F \), gives a rather complete description of the behavior of \( \mathbf{r} = \mathbf{r}_F \) in the vicinity of the null.

Let \( \mathbf{r} = \mathbf{r}_F + \mathbf{\eta} \), \( \mathbf{\eta} \) small.

\[ f_j(\mathbf{r}) = \sum_k \frac{\partial f_j}{\partial r_k} \mathbf{\eta}_k \quad \text{i.e.} \quad \mathbf{f}(\mathbf{r}) = \mathbf{J} \mathbf{\eta} \]

\[ \Rightarrow \quad \mathbf{\dot{r}} = \mathbf{\dot{\eta}} = \mathbf{J} \mathbf{\eta} \]

\( \Rightarrow \) ODE's govern \( \mathbf{r} \approx \mathbf{r}_F \).

Now consider the eigenvalue equation for \( \mathbf{J} \):

\[ \mathbf{J} \mathbf{\eta}_k = \lambda_k \mathbf{\eta}_k \]

Any state \( \mathbf{\eta} \) can be written as a sum of eigenstates,

\[ \mathbf{\eta}(t) = \sum_k \mathbf{C}_k(t) \mathbf{\eta}_k \]

Express both sides of the equation of motion (\( \mathbf{\ddot{r}} \)) in terms of the eigenvectors:

\[ \sum_k \mathbf{C}_k \mathbf{\ddot{\eta}}_k = \sum_k \mathbf{C}_k \lambda_k \mathbf{\dot{\eta}}_k \quad \Rightarrow \quad \mathbf{C}_k = \lambda_k \mathbf{C}_k \]

\[ \Rightarrow \quad \mathbf{C}_k(t) = \mathbf{C}_k(0) e^{\lambda_k t} \quad \forall k \]
INTERPRETATION:

- Each eigenmode evolves separately.
- The mode grows ($\lambda > 0$) or decays ($\lambda < 0$) with time constant $|\lambda|^{-1}$.

Eigenvalues are roots of an $N^{th}$ degree characteristic polynomial. It is therefore possible to have a complex eigenvalue, in which case another eigenvalue will be its complex conjugate:

$$\lambda_1 = \alpha + i\beta \quad (\alpha, \beta \in \mathbb{R}, \beta \neq 0)$$

$$\lambda_2 = \alpha - i\beta$$

The eigenvectors and amplitudes are also complex pairs:

$$\vec{c}_1 = \vec{c} + i\vec{b}, \quad \vec{c}_2 = \vec{c} - i\vec{b}$$

$$\vec{\eta}_1 = \vec{\alpha} + i\vec{\beta}, \quad \vec{\eta}_2 = \vec{\alpha} - i\vec{\beta} \quad (\vec{\alpha}, \vec{\beta} \in \mathbb{R}^N)$$

Since they are eigenvectors, $\vec{\eta}_1 \cdot \vec{\eta}_2 = 0 \Rightarrow \vec{\alpha} \cdot \vec{\beta} = 0$.

WLOG, let $|\alpha| = |\beta| = 1$.

Let $X = \vec{\alpha} \cdot \vec{\eta} = C_1 + C_2$, $Y = \vec{\beta} \cdot \vec{\eta} = i(C_1 - C_2)$

The solution for $X$ & $Y$ is obtained using $\vec{\alpha}$:

$$X = (p_0 + iq_0)e^{(\alpha + i\beta)t} + (p_0 - iq_0)e^{(\alpha - i\beta)t}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = e^{at} \begin{bmatrix} 2p_0\cos bt - 2q_0\sin bt \\ 2q_0\cos bt - 2p_0\sin bt \end{bmatrix}$$

Harmonic motion, with exponential growth or decay, in $xy$-plane!
CHAO5 (BY EXAMPLE)

THE LORENZ EQUATIONS

\[
\begin{align*}
\dot{x} &= \sigma (y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - \beta z
\end{align*}
\]

LORENZ (1963 J. ATM. SCI. 20 321-344) SIMPLIFIED
MODEL OF 2D CONVECTION.

\( \sigma \) — INTENSITY OF CONVECTIVE MOTION
\( \rho \) — TEMP. DIFF. BETWEEN ASCENDING & DESCENDING FLOWS
\( \gamma \) — DISTORTION OF T PROFILE FROM LINEAR.
\( \beta \) — PRANDTL NUMBER \((V/K)\) VISCOITY/CONDUCTIVITY

\( \sigma, \rho, \beta > 0 \)

\[ J(\vec{x}) = \begin{bmatrix}
-\sigma & 0 & 0 \\
\rho - z & -1 & -x \\
y & x & -\beta
\end{bmatrix} \]

THE WIKIPEDIA ARTICLES ON CHAOS, THE LYAPUNOV EXPONENT, AND THE LORENZ ATTRACTION ARE VERY HELPFUL.
Fixed Points:
\[ x = 5(y-x) = 0 \rightarrow y = x \]
\[ \dot{y} = \rho x - \beta y - xz = 0 \rightarrow (\rho - 1 - z)x = 0 \]
\[ \dot{z} = xy - \beta z = 0 \rightarrow \beta z = x^2 \]

Roots:
\[ x = y = z = 0 \text{ ORIGIN} \]
\[ x = y = \pm \sqrt{\beta z}, \quad z = \rho - 1 \]
\[ z = \rho - 1, \quad x = y = \pm \sqrt{\beta (\rho - 1)} \quad "C^+, C^-" \]

\[ \rho = \frac{1}{\beta} x^2 + 1 \]

\[ \text{3 FIXED POINTS} \quad \text{"PITCHFORK"} \quad \text{BIFURCATION} \]
\[ \text{1 FIXED POINT} \]
**Fixed Point at Origin:**

\[ J(\bar{z}) = \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \]

Decouples:

\[ \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sigma & 0 \\ \rho & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ \frac{d}{dt} z = -\beta z \]

**Trick for Solving 2x2:**

\[ M \hat{x} = \lambda \hat{x} \]

\[ \lambda = \frac{\zeta \pm \sqrt{\zeta^2 - 4\Delta}}{2} \]

Where \( \zeta = \text{Tr } M \)

\[ \Delta = \text{Det } M \]

In our case,

\[ \zeta = - (\sigma + 1) \]

\[ \Delta = \sigma \left( 1 - \rho \right) \]

\[ \lambda = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1-\rho)}}{2} \]

\( \beta > 0 \) now, \( \beta < 0 \) and \( \lambda_+ < 0 \) for all \( \rho \).

\( \lambda_- < 0 \) for \( \rho > 1 \) (Saddle)

\( \lambda_+ > 0 \) for \( \rho > 1 \) (Saddle)

\( \lambda_- < 0 \) for \( \rho < 1 \) (Sink)
It turns out that for $\rho < 1$, all trajectories fall into the origin as $t \to \infty$.

So all the interesting physics is for $\rho > 1$.

The behavior of the system changes markedly with $\rho$, going back and forth among a variety of behaviors:
- sometimes periodic orbits
- period doubling
- chaos

We will look at just one classic choice of parameters:

$$\beta = \frac{8}{3}, \quad \sigma = 10, \quad \rho = 28$$

Example using GNU Octave → sensitivity to initial cond.

Like MATLAB,
- vector/matrix oriented
- interpreted
- powerful numerical packages
  - LAPACK (linear algebra)
  - LSODE → includes stiff solver
- etc.
- built-in plotting (GNUplot)
CHAOTIC SYSTEM WORKING DEFINITION:

1. DETERMINISTIC (IN PRINCIPLE, I.E. NOT STOCHASTIC)

2. NO STABLE FIXED POINT OR RUNAWAY CONDITION.

3. SOLUTIONS NOT PERIODIC, NO LIMIT CYCLES.

4. SENSITIVITY TO INITIAL CONDITIONS.

A UNIVERSALLY ACCEPTED DEFINITION OF CHAOS.

DISCUSSION

GIVEN THE ABOVE DEFINITION, IS THE GRAVITATIONAL 3-BODY SYSTEM CHAOTIC?

HOW BIG DOES THE PHASE SPACE HAVE TO BE TO ACCOMMODATE CHAOTIC BEHAVIOR? $R^2$? $R^3$?

LYAPUNOV EXPONENT: CONSIDER 2 NEARBY TRAJECTORIES:

$$|S\vec{r}| = |\vec{r}_2(t) - \vec{r}_1(t)| \approx |S\vec{r}(0)| e^{\lambda t}$$

HOW COULD YOU CALCULATE A "LOCALIZED" LYAPUNOV EXPONENT FROM THE JACOBIAN?