2-POINT BOUNDARY VALUE PROBLEMS (NR CH. 17)

So far, we have been solving initial value problems. Suppose instead we are given

\[ \frac{dx}{dt} = f(x, t) \quad \text{--- \# FIRST-ORDER EQUATIONS} \]

The two common approaches to this problem are:

1. SHOOTING (NR §§ 17.1-17.2)
2. RELAXATION (NR § 17.3)
**Shooting — Simplest Realization**

**Suppose** \( \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{f} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \)

**Given:**
- \( x(t_1) = x_1 \)
- \( x(t_2) = x_2 \)

**Find:**
- \( x(t) \), \( y(t) \)

**Let** \( \mathbf{R} \) **be the propagator from** \( t_1 \) **to** \( t_2 \):

\[
\mathbf{R} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}
\]

**Q:** How would you implement \( \mathbf{R}() \) in Octave?

**Consider the problem**

\[
\mathbf{R}_x \begin{bmatrix} x_1 \\ x \end{bmatrix} = x_2,
\]

\( x \) **unknown**

If we can solve this problem, we'll have

\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

This can be rewritten as a root-finding problem:

\[
G(y) = \mathbf{R}_x \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) - x_2
\]

\( G(y_1) = 0 \)

**Q:** How would you implement \( G(y) \) in Octave?
ROOT FINDING

GIVEN SOME \( G(y) \), FIND \( y \) SATISFYING \( G(y) = 0 \)

BRACKETING:

IF \( G(a) \) \( G(b) < 0 \), \( a < b \), THEN \( a < y < b \).

\[ G(y) \]
\[ G(a) \]
\[ G(b) \]
\[ a \quad b \]
\[ y \]

BISECTION ALGORITHM:

1. LET \( y' = \frac{1}{2}(a+b) \)
2. IF \( G(y') = 0 \) THEN \( \text{DONE: } G(y') \approx 0 \)
3. ELSE IF \( G(y') \cdot G(a) > 0 \)
   3.1. \( b = y' \)
   3.2. \( a = y' \)
   3.3. \( \text{GO TO 1} \)
4. ELSE \( b - a < \text{tol} \)
5. \( \text{DONE} \)
HOW FAST DOES BISECTION CONVERGE?

The error is $\varepsilon = b - a$.

$$
\varepsilon_{n+1} = \frac{1}{2} \varepsilon_n
$$

$$
\varepsilon_n = \left(\frac{1}{2}\right)^n \varepsilon_0
$$

WHERE $n =$ # I TERATIONS, $\varepsilon_0 =$ I NITIAL SIZE OF BRACKET, $b - a$. 
**Newton's Method**

Begin with a guess, $y_1$.

**Iteration Formula:**

$$G'(y_n) = \frac{G(y_n)}{y_n - y_{n+1}}$$

$$y_{n+1} = y_n - \frac{G(y_n)}{G'(y_n)}$$
CONVERGENCE OF NEWTON'S METHOD

EXPAND ABOUT THE ROOT AT $y = y_R$:

$$G(y) = G(y_R) + (y - y_R) G'_R + \frac{1}{2} (y - y_R)^2 G''_R$$

WHERE $G'_R \equiv G'(y_R)$

$G''_R \equiv G''(y_R)$

NOW $G'(y) = G'_R + (y - y_R) G''_R$

OUR ITERATION FORMULA NOW BECOMES:

$$y_{n+1} = y_n - \frac{G(y_n)}{G'(y_n)}$$

$$y_{n+1} = y_n - \frac{(y_n - y_R) G'_R + \frac{1}{2} (y_n - y_R)^2 G''_R}{G'_R + (y_n - y_R) G''_R}$$

SUBTRACT $y_R$ FROM BOTH SIDES.

LET $\epsilon_n = y_n - y_R$
\[ \varepsilon_{n+1} = \varepsilon_n - \frac{\varepsilon_n G_R' + \frac{1}{2} \varepsilon_n^2 G_R''}{G_R' + \varepsilon_n G_R''} \]

\[ = \frac{\left(\varepsilon_n G_R' + \varepsilon_n^2 G_R''\right) - \left(\varepsilon_n G_R' + \frac{1}{2} \varepsilon_n^2 G_R''\right)}{G_R' + \varepsilon_n G_R''} \]

\[ = \frac{\frac{1}{2} \varepsilon_n^2 G_R''}{G_R' + \varepsilon_n G_R''} \]

For small \( \varepsilon_n \) \( \xrightarrow{\text{compared to what?}} \)

\[ \varepsilon_{n+1} \approx \frac{1}{2} \varepsilon_n \frac{G_R''}{G_R'} \]

\( \text{Q: Is this faster or slower than bisection?} \)
Newton can be fooled:

If we hit small \( g' \) can be deflected far away.

Can get into a limit cycle.
## TRADE-OFFS

<table>
<thead>
<tr>
<th>NEWTON</th>
<th>BISECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>FAST</td>
<td>SLOW ( (\varepsilon_n \sim 2^{-n}) )</td>
</tr>
<tr>
<td>FRAGILE</td>
<td>ROBUST ( (\text{GUARANTEED CONVERGENCE}) )</td>
</tr>
<tr>
<td>REQUIRES GOOD INITIAL GUESS</td>
<td>REQUIRES BRACKET</td>
</tr>
</tbody>
</table>

### NEXT

- EXAMPLE
- HIGHER - DIMENSIONAL ROOT FINDING
  - SHOOTING IN SYSTEMS WHERE WE NEED TO SOLVE FOR MULTIPLE UNKNOWN PARAMETERS TO COMPLETE THE INITIAL CONDITION.