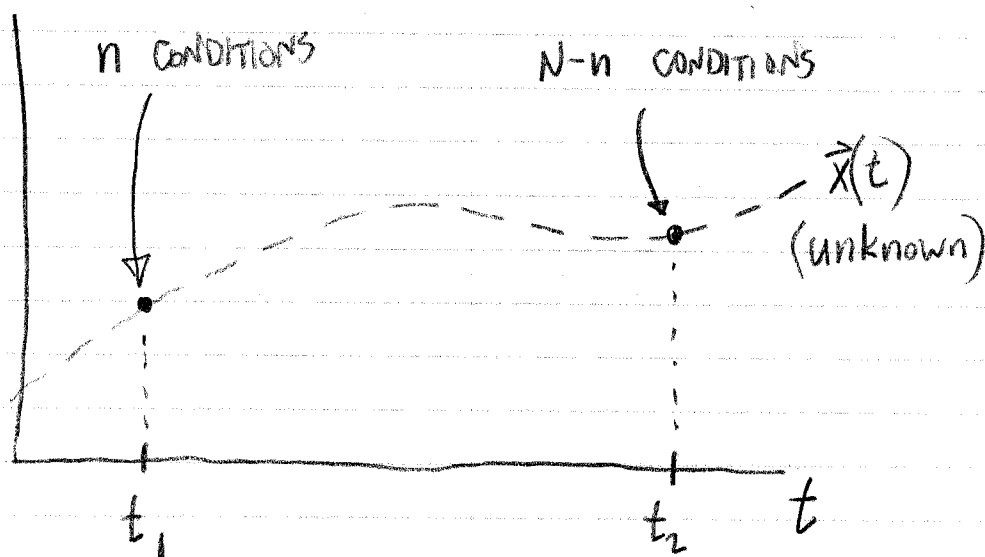


2-POINT BOUNDARY VALUE PROBLEMS (NR CH. 17)

SO FAR, WE HAVE BEEN SOLVING INITIAL VALUE PROBLEMS. SUPPOSE INSTEAD WE ARE GIVEN

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t) \quad \text{--- } N \text{ FIRST-ORDER EQUATIONS}$$



THE TWO COMMON APPROACHES TO THIS PROBLEM ARE

- ① SHOOTING (NR §§ 17.1-17.2)
- ② RELAXATION (NR § 17.3)

SHOOTING — SIMPLEST REALIZATION

SUPPOSE $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{f} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right)$

GIVEN: $x(t_1) = x_1$
 $x(t_2) = x_2$

FIND: $x(t), y(t)$

LET \vec{R} BE THE PROPAGATOR FROM t_1 TO t_2 :

$$\vec{R} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Q: HOW WOULD YOU IMPLEMENT $\vec{R}()$ IN OCTAVE?

CONSIDER THE PROBLEM

$$R_x \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) = x_2.$$

UNKNOW

IF WE CAN SOLVE THIS PROBLEM, WE'LL HAVE

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

THIS CAN BE REWRITTEN AS A ROOT-FINDING PROBLEM:

$$G(y) \equiv R_x \left(\begin{bmatrix} x_1 \\ y \end{bmatrix} \right) - x_2$$

$$G(y_1) = 0$$

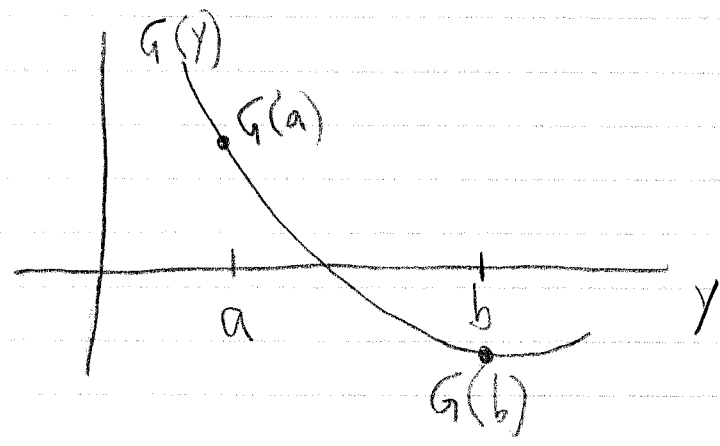
Q: HOW WOULD YOU IMPLEMENT $G(y)$ IN OCTAVE?

ROOT FINDING

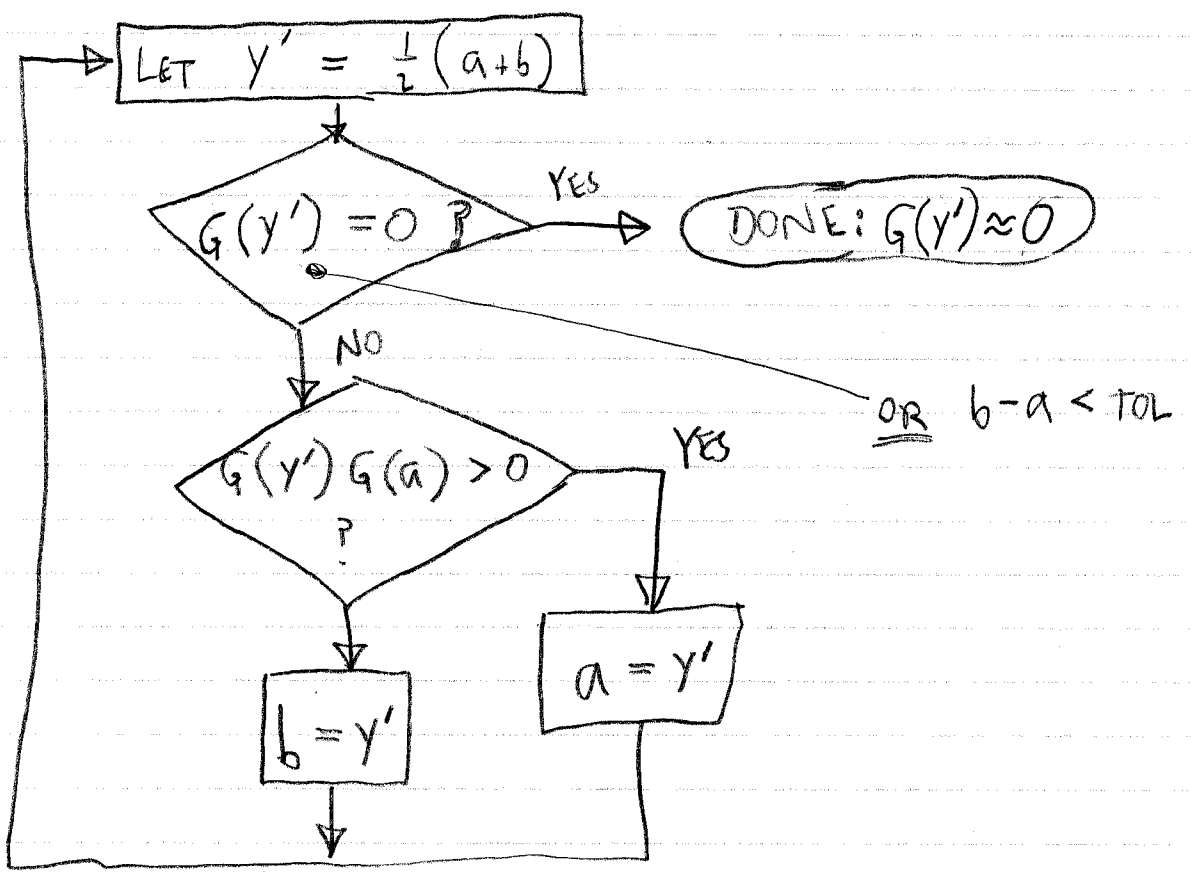
GIVEN SOME $G(y)$, FIND y SATISFYING $G(y) = 0$

BRACKETING:

IF $G(a)G(b) < 0$, $a < b$, THEN $a < y < b$.



BISECTION ALGORITHM:



(4)

HOW FAST DOES BISECTION CONVERGE?

THE ERROR IS $\epsilon = b - a$.

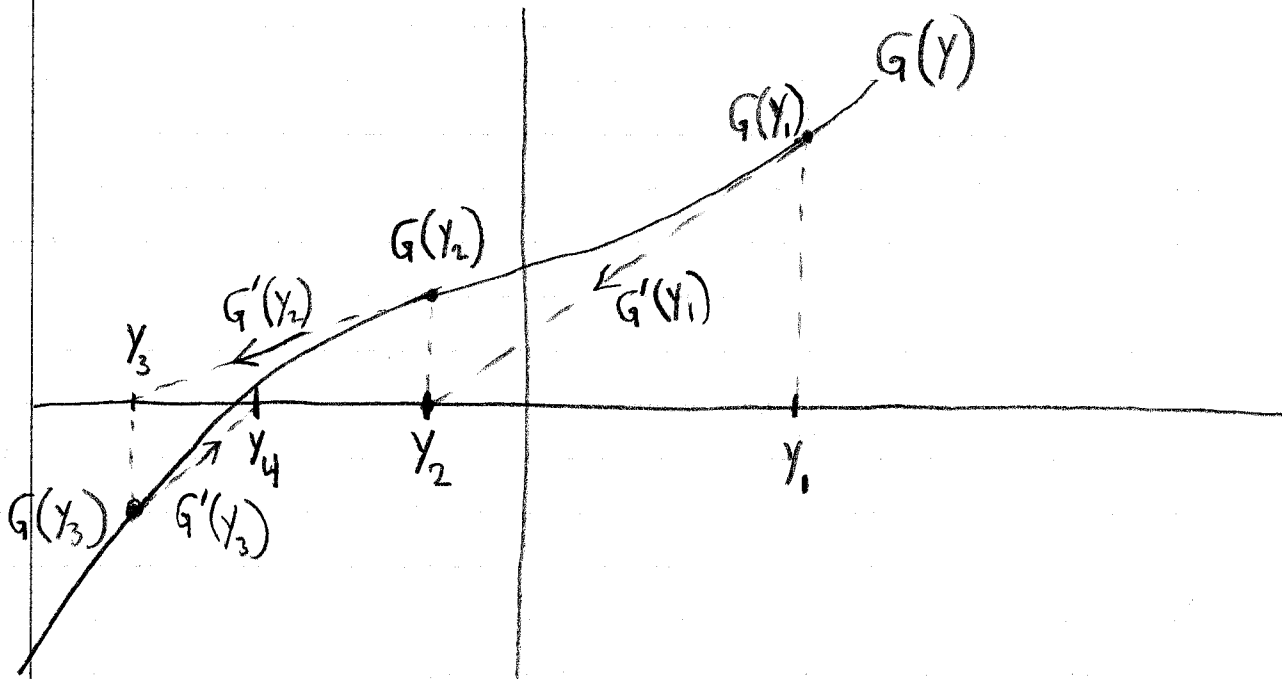
$$\epsilon_{n+1} = \frac{1}{2} \epsilon_n$$

$$\epsilon_n = \left(\frac{1}{2}\right)^n \epsilon_0$$

WHERE $n = \#$ ITERATIONS, $\epsilon_0 =$ INITIAL SIZE OF BRACKET, $b - a$.

NEWTON'S METHOD

BEGIN WITH A GUESS, y_1 .



ITERATION FORMULA:

$$G'(y_n) = \frac{G(y_n)}{y_n - y_{n+1}}$$

$$y_{n+1} = y_n - \frac{G(y_n)}{G'(y_n)}$$

(6)

CONVERGENCE OF NEWTON'S METHOD

EXPAND ABOUT THE ROOT AT $y = y_R$:

$$G(y) = \cancel{G(y_R)} + (y - y_R) G'_R + \frac{1}{2} (y - y_R)^2 G''_R$$

$$\text{WHERE } G'_R \equiv G'(y_R)$$

$$G''_R \equiv G''(y_R)$$

$$\text{NOW, } G'(y) = G'_R + (y - y_R) G''_R$$

OUR ITERATION FORMULA NOW BECOMES:

$$y_{n+1} = y_n - \frac{G(y_n)}{G'(y_n)}$$

$$= y_n - \frac{(y_n - y_R) G'_R + \frac{1}{2} (y_n - y_R)^2 G''_R}{G'_R + (y_n - y_R) G''_R}$$

• SUBTRACT y_R FROM BOTH SIDES.

• LET $\epsilon_n = y_n - y_R$

(7)

$$\epsilon_{n+1} = \epsilon_n - \frac{\epsilon_n G_R' + \frac{1}{2} \epsilon_n^2 G_R''}{G_R' + \epsilon_n G_R''}$$

$$= \frac{(\epsilon_n G_R' + \epsilon_n^2 G_R'') - (\epsilon_n G_R' + \frac{1}{2} \epsilon_n^2 G_R'')}{G_R' + \epsilon_n G_R''}$$

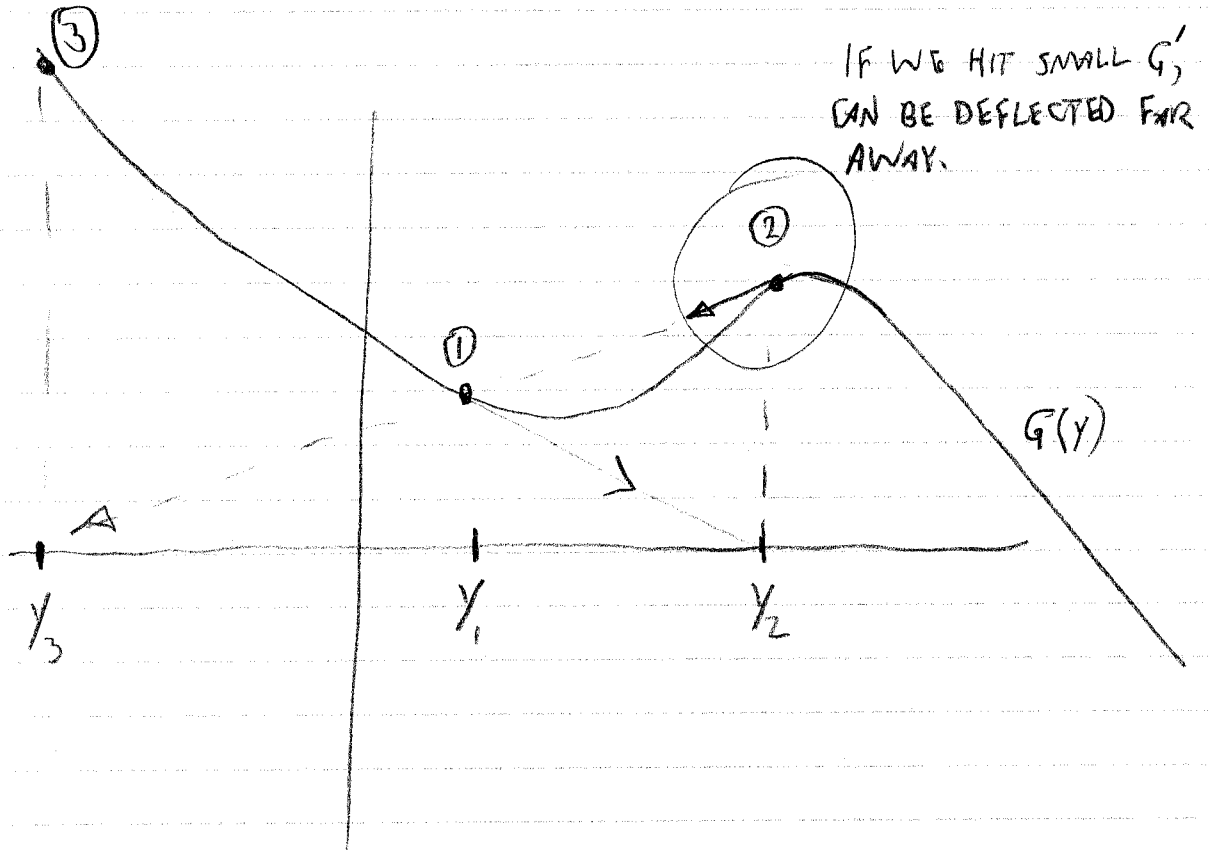
$$= \frac{\frac{1}{2} \epsilon_n^2 G_R''}{G_R' + \epsilon_n G_R''}$$

FOR SMALL ϵ_n \longrightarrow COMPARED TO WHAT?

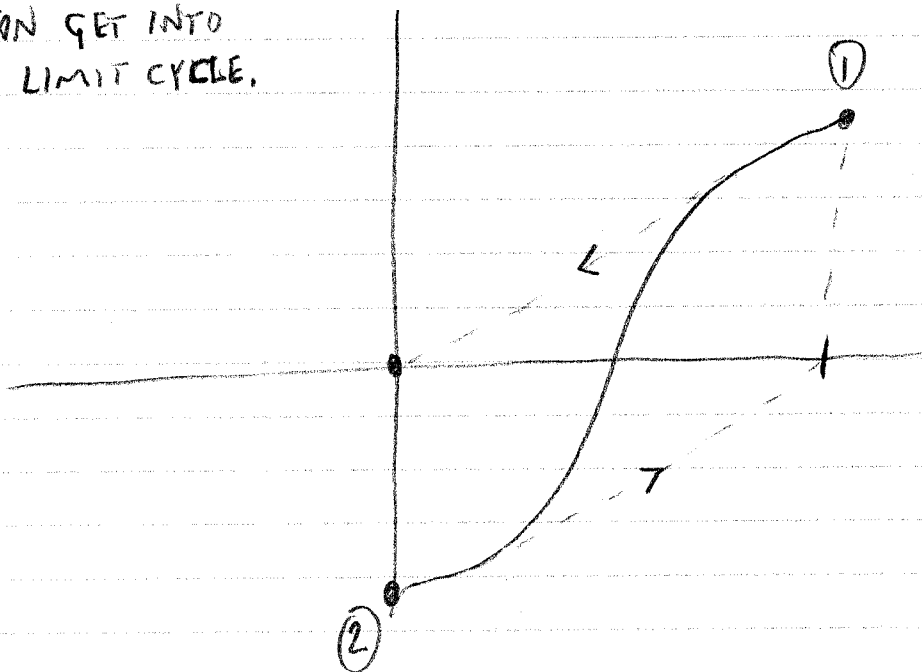
$$\epsilon_{n+1} \approx \frac{1}{2} \epsilon_n^2 \frac{G_R''}{G_R'}$$

Q: IS THIS FASTER OR SLOWER THAN BISECTION?

NEWTON CAN BE FOOLED:



CAN GET INTO A LIMIT CYCLE.



TRADE-OFFS

NEWTON	BISECTION
FAST	SLOW ($\epsilon_n \sim 2^{-n}$)
FRAGILE	ROBUST (GUARANTEED CONVERGENCE)
REQUIRES GOOD INITIAL GUESS	REQUIRES BRACKET

NEXT

- EXAMPLE
- HIGHER-DIMENSIONAL ROOT FINDING
 - SHOOTING IN SYSTEMS WHERE WE NEED TO SOLVE FOR MULTIPLE UNKNOWN PARAMETERS TO COMPLETE THE INITIAL CONDITION.