"TRICK SHOOTING"

I. HIGH-DIMENSIONAL (>2) SYSTEMS OF ODE's:

NR § 17.1 shoot()

NR § 9.6 NEWTON'S METHOD, Minewt() FOR SYSTEMS OF NONLINEAR EQ's

THE SHOOTING PROBLEM IN N DIMENSIONS:

\[
\frac{d}{dt} \vec{y} = \vec{f}(\vec{y}, t), \quad \vec{y} \in \mathbb{R}^N
\]

INTEGRATE FROM \( \vec{y}^0 = \vec{y}(t_0) \) TO \( \vec{y}' = \vec{y}(t_1) \)

GIVEN BOUNDARY CONDITIONS:

\[
y_1^0, y_2^0, y_3^0, \ldots, y_n^0; y_{n+1}^i, y_{n+2}^i, \ldots y_N^i
\]

WE HAVE A PROCEDURE \( \vec{R} \) THAT INTEGRATES:

\[
\vec{R}(\vec{y}^0) = \vec{y}'
\]

CONSIDER N-th COMPONENTS TO SOLVE FOR t=0 UNKNOWNS:

\[
\vec{R}_n(\vec{y}^0) = y_n, \quad n+1 \leq n \leq N
\]

N-n EQ's, N-n UNKNOWNS.
Rewrite as
\[ g^* \left( y_{n+1}, \ldots, y_N \right) \equiv R_k(y^0) - y_k' = 0 \]
\[ \bar{g} \left( \bar{q} \right) = 0, \quad \bar{g}, \bar{q} \in \mathbb{R}^M, \quad M = N - m \]

\Rightarrow \text{MULTIDIM. ROOT FINDING.}

[And when we're done, we run \( \bar{R} \) once to integrate from \( y_0 \) (known) to \( y_N \).]

The hard part is solving \( \bar{A} \).

Can we bracket the root? Consider \( M = 2 \):

\[ \begin{align*}
&g_1 = 0 \\
&g_2 = 0 \\
&g_1 = 0 \\
&g_2 = 0
\end{align*} \]

We can only sample individual points, and no number of points can be used to enclose a (null) point in 2D.

No, cannot bracket.
Since we cannot bracket roots in $\mathbb{R}^M$ for $M > 1$ dimensions, there is no analogue to bisection and no guarantee of convergence.

All we can do is use slopes to search for roots. Recall Newton's method:

$$\tilde{b}^* = b - \frac{G(\tilde{b})}{G'(\tilde{b})}$$

We seek a multi-dimensional analogue...

Want to satisfy

$$G(\tilde{b}^*) = 0$$

Suppose

$$\tilde{b}^* = \tilde{b} + \delta \tilde{b}$$

$$\tilde{G}(\tilde{b}^*) = \tilde{G}(\tilde{b}) + \sum_i \frac{\partial \tilde{G}}{\partial \tilde{b}_i} \delta \tilde{b}_i + O(\delta \tilde{b}^2)$$

Or in component form,

$$G_j(\tilde{b}^*) = \sum_i \frac{\partial \tilde{G}_j}{\partial \tilde{b}_i} \delta \tilde{b}_i + G_j(\tilde{b})$$

The Jacobian matrix $J$ is

If we want to satisfy $\star$,

$$J \delta \tilde{b} + \tilde{G}(\tilde{b}) = 0$$
\[ g' = -J^{-1} \hat{G}(\hat{\theta}) \]

\[ \hat{\theta}^* = \hat{\theta} - J^{-1} \hat{G}(\hat{\theta}) \]

This is Newton's method in N-dimensions.

NR § 9.6, mention.

This is all we need to solve the general 2-point B.V. problem for an N-dimensional ODE.

Q: What is the relationship between
- N = size of phase space
- M = # dimensions of \( \hat{G}(\hat{\theta}) \)

A: M is at most N-1, but if we were free to choose the direction of integration,

\[ M \leq \frac{N}{2}. \]
NONLINEAR EQUATION SOLVERS:

\textsc{octave} \texttt{fsolve} \quad (\textsc{minpack} \textsc{hybrd})^*

\textsc{matlab} \texttt{fsolve} \quad (\textsc{multiple algorithms})

Another strategy for solving
\[ \vec{G}(\vec{b}) = 0 \]

is to let
\[ f(\vec{b}) = | \vec{G} |^2. \]

This is now recast as a problem of minimizing
a scalar function of several variables.

Minimization is a topic of broad interest with many applications.

* Powell hybrid method, a Newton-Broyden solver.