

HYDRODYNAMICS OF IDEAL GASES --- SOUND WAVES IN 1D

NAVIER-STOKES:

$$\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial x} - \frac{1}{\rho} \nabla P \leftarrow (\text{DISCUSS THE PHYSICS!})$$

CONTINUITY:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) = -v \frac{\partial \rho}{\partial x} - \rho \frac{\partial v}{\partial x} \leftarrow$$

ADIABATIC:

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma, \quad \gamma = \frac{f+2}{f} \in \left\{ \overset{\text{MONATOMIC}}{\frac{5}{3}}, \overset{\text{DIATOMIC}}{\frac{7}{5}}, \overset{\text{POLYATOMIC}}{\frac{4}{3}} \right\}$$

$$\text{LET } P = P_0 (1+\eta)$$

$$\rho = \rho_0 (1+\eta)^{1/\gamma}$$

$$\boxed{\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial x} - \frac{P_0}{\rho_0} (1+\eta)^{-1/\gamma} \frac{\partial \eta}{\partial x}} \quad (1)$$

$$\frac{\partial \rho}{\partial t} = \frac{\rho_0}{\gamma} (1+\eta)^{1/\gamma-1} \frac{\partial \eta}{\partial t} = -v \frac{\rho_0}{\gamma} (1+\eta)^{1/\gamma-1} \frac{\partial \eta}{\partial x} - \rho_0 (1+\eta)^{1/\gamma} \frac{\partial v}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial \eta}{\partial t} = -v \frac{\partial \eta}{\partial x} - \gamma (1+\eta) \frac{\partial v}{\partial x}} \quad (2)$$

LOW-AMPLITUDE WAVES:

- FIRST ORDER IN v, η

$$\left. \frac{\partial \eta}{\partial t} = -\gamma \frac{\partial v}{\partial x} \right] \quad (3)$$

$$\left. \frac{\partial v}{\partial t} = -\frac{P_0}{\rho_0} \frac{\partial \eta}{\partial x} \right] \quad (4)$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\gamma \frac{\partial^2 v}{\partial x \partial t} = -\gamma \frac{\partial}{\partial x} \left(-\frac{P_0}{\rho_0} \frac{\partial \eta}{\partial x} \right)$$

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} = \gamma \frac{P_0}{\rho_0} \frac{\partial^2 \eta}{\partial x^2}}$$

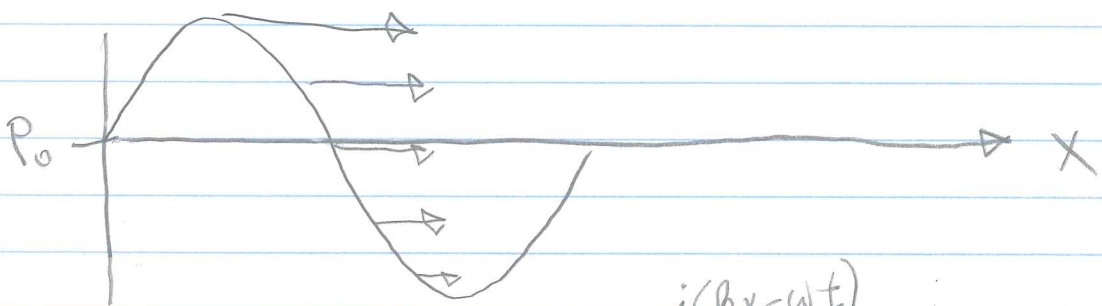
c^2

THE LINEARIZED EQUATION CONTAINS 2 HINTS ABOUT THE NONLINEAR SYSTEM IT CAME FROM:

I. SINCE $c = \gamma \frac{P}{\rho}$, THE PROPAGATION SPEED EVIDENTLY VARIES WITH P & ρ. IF WE TAKE

$$c = \gamma \frac{P}{\rho} = \gamma \frac{P_0}{\rho_0} (1 + \eta)^{1 - 1/\gamma}$$

APPARENTLY, SINCE $1 - 1/\gamma > 0$, c INCREASES AT THE CREST OF A PRESSURE WAVE, AND DECREASES IN THE TROUGH:



II. IF WE SUBSTITUTE $\eta = e^{i(kx - \omega t)}$ INTO THE FIRST ORDER EQUATIONS, WE FIND

$$v = \frac{\omega}{k} \eta$$

WHICH MEANS v IS IN PHASE WITH η. NOW RECALL WE ASSUMED v IS SMALL; WHEN v IS NOT SMALL, THE SOUND SPEED c MUST BE MEASURED RELATIVE TO THE REST FRAME OF THE GAS. IN THE LAB FRAME, THEN, THE SPEED OF THE WAVE IS c + v. THUS, THE CRESTS SHOULD MOVE FASTER THAN THE TROUGHS, AS WE FOUND IN I.

NONE OF THIS IS PREDICTED BY THE LINEARIZED EQNS., OF WHICH $e^{i(kx - \omega t)}$ IS AN EXACT SOLN. BUT WE INFER THAT THE FULL NONLINEAR EQNS. SHOULD PRODUCE SHOCKS.

THE EXAMPLE (WAVE 2.C) INCLUDES BOTH THE LINEAR & NONLINEAR SOUND WAVES.

#DEFINE PDE pde_1 \rightarrow EQS. 3 & 4 (LINEARIZED)
 #DEFINE PDE pde_2 \rightarrow EQS. 1 & 2 (NONLINEAR)

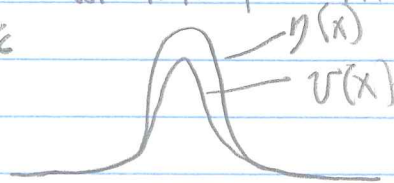
THINGS TO LOOK AT:

- STATE VECTOR = q , $q[0][i] = \eta_i$, $q[1][i] = v_i$
- GENERAL PURPOSE INTEGRATOR, `rk2plus()`
- SAFETY FACTOR (float safety in `main()`)
- VISUALIZATION STRATEGY
 - GENERATE TIME SEQUENCE OF PLOTS
 - SAVE AS NUMBERED IMAGE FILES
 - SUCK INTO A MOVIE MAKING UTILITY (I USE QUICKTIME PRO)

QUICKTIME MOVIES ARE ON THE WEB WITH THE EXAMPLE.

- AS PREDICTED, THE NONLINEAR EQ'S CAUSE THE PROPAGATING PULSE TO STEEPEN INTO A SHOCK.

Q: RIGHTWARD PROPAGATING INITIAL CONDITION LOOKS LIKE



WHAT DOES LEFT-PROPAGATING PULSE LOOK LIKE?

A: $v \rightarrow -v$:

