

INTEGRAL EQUATIONS (INVERSE PROBLEMS)

A GOOD BOOK ON THE SUBJECT (BUT OUT OF PRINT, HARD TO FIND, AND PROHIBITIVELY EXPENSIVE EVEN IF YOU FIND IT) IS INVERSE PROBLEMS IN ASTRONOMY BY IAN CRAIG & JOHN BROWN (HILGER, 1986).

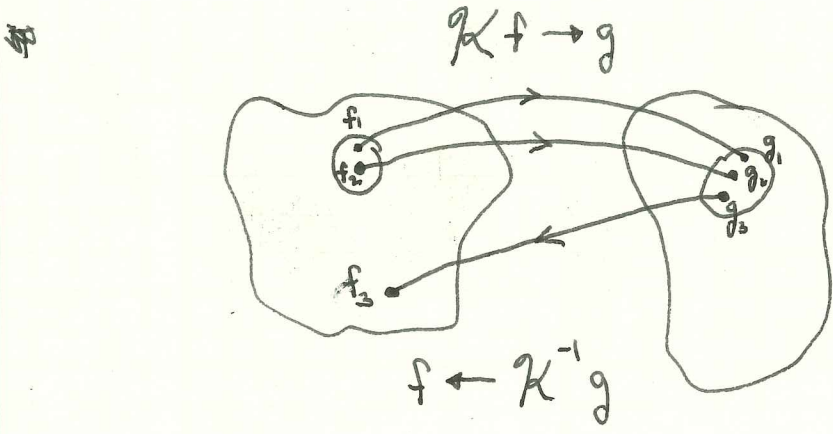
(SEE ALSO RILEY, HOBSON & BENICE CH.23)

WE'LL BEGIN BY DISCUSSING THE COVER ILLUSTRATION FROM CRAIG & BROWN.

CONSIDER AN INTEGRAL OPERATOR \mathcal{K} . THESE TAKE THE FORM

$$\mathcal{K} f \equiv \int \underset{\substack{\uparrow \\ \text{KERNEL}}}{K} f(x) dx$$

WHERE THE INTEGRAL MAY BE DEFINITE OR INDEFINITE, $f(x)$ IS A FUNCTION, x IS SCALAR OR VECTOR IN THE DOMAIN.



THE GENERAL PROBLEM IS GIVEN $g = \mathcal{K} f$, FIND f .

AS THE ILLUSTRATION SHOWS, THIS IS SOMETIMES HARDER THAN IT LOOKS, EVEN IF THE OPERATOR \mathcal{K} IS LINEAR (THE ONLY TYPE CONSIDERED BY CRAIG & BROWN).

GENERAL FORM OF THE LINEAR INTEGRAL EQUATION:

$$\alpha(x) f(x) + \int_{a(x)}^{b(x)} K(x, x') f(x') dx' = g(x)$$

\uparrow known \uparrow \uparrow KERNEL (KNOWN) \uparrow UNKNOWN \uparrow known

CLASSIFICATION

1st KIND: $\alpha(x) = 0$

2nd KIND: $\alpha(x) \neq 0$

FREDHOLM: a, b CONSTANTS

VOLTERRA: $a = \text{CONSTANT}$
 $b = x$

EXAMPLES...

ALL ELSE BEING EQUAL, EQUATIONS OF THE SECOND KIND ARE EASIER.

CONSIDER:

$$g(x) = f(x) + \int_a^b K(x,y) f(y) dy$$

IF K IS SUFFICIENTLY SMALL ($\int_a^b K(x,y) dy \ll 1 \forall x$), THE FOLLOWING

ITERATIVE METHOD CONVERGES:

$$f_0(x) = g(x)$$

$$f_1(x) = g(x) - \int_a^b K(x,y) f_0(y) dy$$

$$\vdots$$

$$f_n(x) = g(x) - \int_a^b K(x,y) f_{n-1}(y) dy$$

RILEY, HOBSON & BENGE CALL THIS A "NEUMANN SERIES".

THINK OF OUR OPERATOR AS A MATRIX:

$$\mathcal{K} \vec{f} = \vec{g}, \quad \mathcal{K}^{-1} \vec{g} = \vec{f}$$

$$\mathcal{K} = \underset{\substack{\uparrow \\ \text{IDENTITY}}}{\mathbb{I}} + K$$

IF THE ~~THE~~ OFF-DIAGONAL TERMS IN K ARE SMALL, THEN \mathcal{K} IS DIAGONAL DOMINANT. SUCH OPERATORS ARE WELL-CONDITIONED, IN THE SENSE THAT THE INVERSE ~~IS NOT~~ PROVIDES AN ANSWER THAT IS NOT TOO SENSITIVE TO SMALL CHANGES IN \vec{g} .



SPECTROMETER DECONVOLUTION PROBLEM

$$g(\lambda) = \alpha f(\lambda) + \beta \int_{-\infty}^{\infty} K(\lambda - \lambda') f(\lambda') d\lambda'$$

WHERE K REPRESENTS THE SCATTERING OF LIGHT BY THE DIFFRACTION GRATINGS; NORMALIZATION:

$$\int_{-\infty}^{\infty} K(\lambda) d\lambda = 1$$

$$\alpha + \beta = 1$$

SOLVE BY NEUMANN SERIES:

$$\text{LET } f_0(\lambda) = g(\lambda)$$

$$f_1(\lambda) = \frac{1}{\alpha} g(\lambda) - \frac{\beta}{\alpha} \int_{-\infty}^{\infty} K(\lambda - \lambda') f_0(\lambda') d\lambda'$$

$$\vdots$$

$$f_n(\lambda) = \frac{1}{\alpha} g(\lambda) - \frac{\beta}{\alpha} \int_{-\infty}^{\infty} K(\lambda - \lambda') f_{n-1}(\lambda') d\lambda'$$

COMPUTATIONAL NOTE:

NOTE THAT ALL STEPS CAN BE DONE QUICKLY IN FOURIER SPACE!

VOLTERRA EQUATIONS ARE EASIER TO SOLVE THAN FREDHOLM.

CONSIDER THE VOLTERRA EDN. OF THE 1ST KIND:

$$\int_a^x K(x,y) f(y) dy = g(x)$$

DIFFERENTIATE WITH RESPECT TO x :

$$K(x,x) f(x) + \int_a^x \frac{\partial K(x,y)}{\partial x} f(y) dy = \frac{dg}{dx}$$

THIS IS NOW A VOLTERRA EDN. OF THE SECOND KIND.

WE COULD TAKE MORE DERIVATIVES... IF THE INTEGRAL PORTION EVENTUALLY BECOMES IGNOREABLE, THEN WE HAVE AN INHOMOGENEOUS ODE.

FROM A MATRIX POINT OF VIEW,

$$\mathcal{K} \vec{f} = \vec{g},$$

\mathcal{K} IS LOWER TRIANGULAR, HENCE EASY TO SOLVE BY ROW REDUCTION.

DECONVOLUTION (FREDHOLM, 1st KIND)

$$g(x) = \int_{-\infty}^{\infty} K(x-x') f(x) dx$$

↑
UNKNOWN

FORMAL SOLUTION:

$$\tilde{g} = \tilde{K} \tilde{f}$$

$$\Rightarrow \boxed{\tilde{f} = \tilde{g} / \tilde{K}}$$

★ LOOK AT deconvolution / deconvolution-natf.m

→ NULL SPACE
 FOR "LARGE" h , $\tilde{K}(h) \approx 0$
 $\Rightarrow K * e^{ihx} \approx 0 e^{ihx}$

→ FORMAL SOLN. FAILS FOR LARGE h

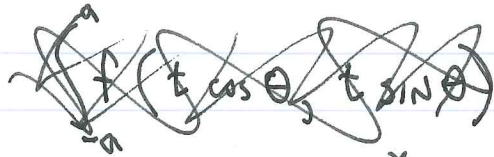
★ LOOK AT cutoff.m, deconvolution.m

→ REGULARIZATION

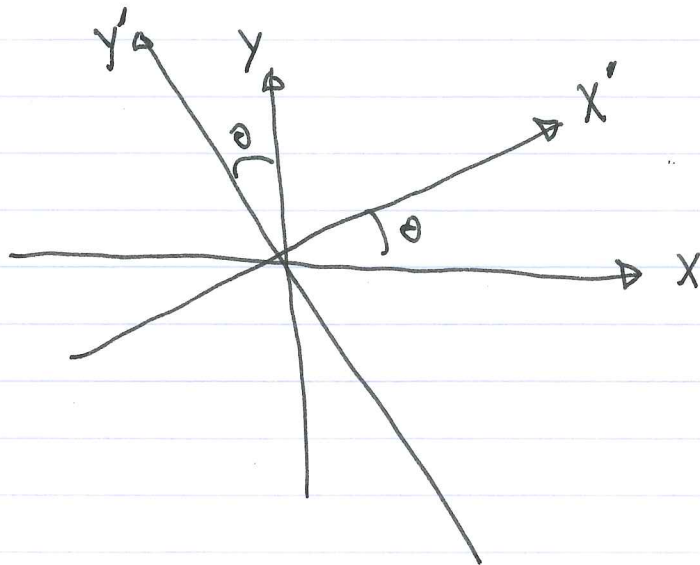
TOMOGRAPHY

UNKNOWN: $f(x,y)$ ("OBJECT")

OBSERVABLES ARE PROJECTIONS:



$$g_{\theta}(X') = \int_{-a}^a f(\underbrace{x' \cos \theta - y' \sin \theta}_x, \underbrace{x' \sin \theta + y' \cos \theta}_y) dy'$$



$g_{\theta}(X')$ is called the "RADON TRANSFORM" of f .

AND MY 2008 OPTIC PRESENTATION
LOOK AT EXAMPLES UNDER DECONVOLUTION.

CONSIDER THE EXAMPLES WITH 36 LOOK ANGLES.
INFORMATION CONTENT IS $36N = 36 \times 628$ ~~WORDS~~ $\sim 1/4$
THE CONTENT OF THE ORIGINAL IMAGE. NOT BAD!

TOMOGRAPHY — ADDITIONAL INFO

TRANSMISSION TOMOGRAPHY: (E.G. MEDICAL CAT SCAN)

TRANSMISSION $T = e^{-\tau}$, $\tau = \text{OPTICAL DEPTH} \equiv -\ln T$

TRANSMISSION THROUGH 2 OBSTRUCTIONS:

$$T_{\text{TOT}} = T_1 T_2 = e^{-(\tau_1 + \tau_2)}$$

$$\text{SO } \tau_{\text{TOT}} = \tau_1 + \tau_2$$

• SO TOMOGRAPHY BY TRANSMISSION CAN BE WRITTEN AS A LINEAR PROBLEM USING THE RADON TRANSFORM.

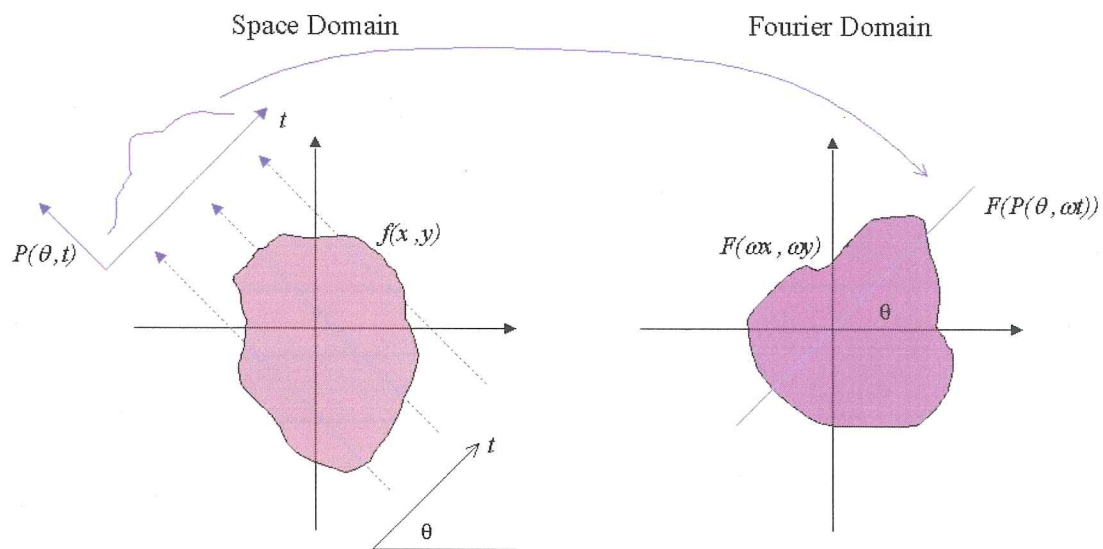
• MORE GENERALLY, THE OBJECT IS DESCRIBED

BY AN ABSORBANCE $a(\vec{r})$, } DISCUSS UNITS OF $a(\vec{r})$.

$$\tau_{\mathcal{L}} = \int_{\mathcal{L}} a \, ds$$

WHERE \mathcal{L} IS A PARTICULAR LINE OF SIGHT.

Fourier Slice Theorem



Projection under angle θ

equals

slice under θ in fourier domain

SEE PRINCIPLES OF COMPUTERIZED TOMOGRAPHY BY KAK & SLANEY

WE WILL NOW DO A NONLINEAR INTEGRAL EQUATION!

LIDAR

CONSIDER A BEAM OF LIGHT PROPAGATING THROUGH SPACE:



POWER IN THE OUTGOING BEAM AT $r+dr$:

$$p(r+dr) = p(r) \left[1 - \underset{\substack{\uparrow \\ \text{EXTINCTION}}}{\sigma(r)} dr \right]$$

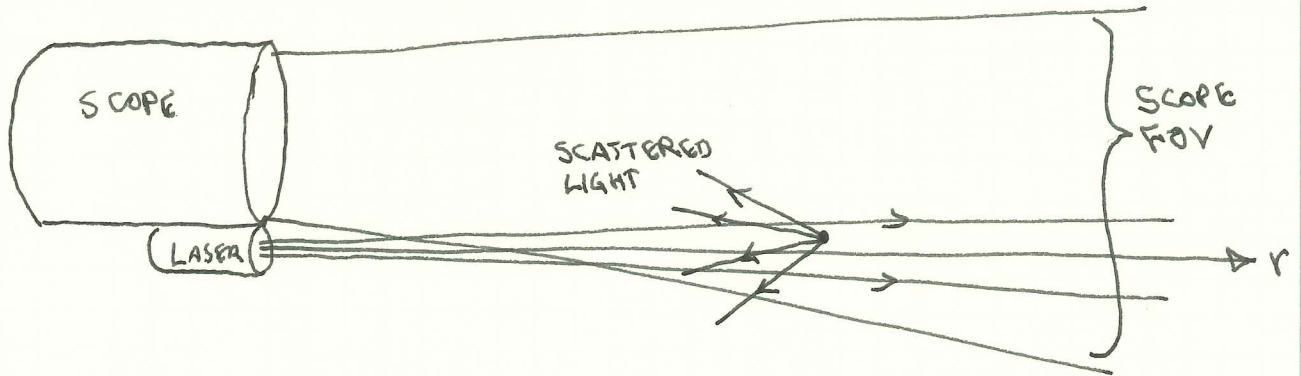
$$\Rightarrow \frac{1}{p} \frac{dp}{dr} = \cancel{p(r)} \cancel{dr} - \sigma(r)$$

$$\Rightarrow \ln(p/p_0) = -\int_0^r \sigma(r') dr'$$

WHERE $p_0 \equiv p(0)$

$$\therefore p(r) = p_0 e^{-\int_0^r \sigma(r') dr'}$$

NOW SUPPOSE THAT THE SOURCE IS ATTACHED TO A TELESCOPE WHOSE FIELD OF VIEW IS WIDER THAN THE BEAM:



THE LASER IS PULSED AT TIME $t = 0$.

AT RANGE r , THERE IS SOME DENSITY OF SCATTERERS SUCH THAT, OVER INTERVAL $r \dots r + dr$, A FRACTION βdr OF THE LIGHT IS SCATTERED.

THE SCATTERED LIGHT FROM RANGE r IS DETECTED AT THE TELESCOPE AT TIME t ,

$$ct = 2r. \quad (\text{TIME} \iff \text{RANGE})$$

WE NOW RECEIVE A SIGNAL OF INTENSITY $P(r)$, NOT TO BE CONFUSED WITH INCIDENT INTENSITY $p(r)$:

$$P(r) = \frac{AK}{r^2} \underbrace{p(r)}_{\text{INCIDENT POWER AT RANGE } r} \underbrace{\beta(r)}_{\text{SCATTERING FRACTION PER PATH LENGTH}} \underbrace{e^{-2 \int_0^r \sigma(r') dr'}}_{\text{ATTENUATION ALONG RETURN TRIP}}$$

A = TELESCOPE COLLECTING AREA

$\left(\frac{K}{r^2}\right)$ = SCATTERING GEOMETRY FACTOR, FOR EXAMPLE, $K = 1/4\pi$ FOR ISOTROPIC SCATTERING.

PLUGGING IN OUR PREVIOUS RESULT FOR $p(r)$,

$$P(r) = \frac{AK}{r^2} p_0 \beta(r) e^{-2 \int_0^r \sigma(r') dr'}$$

THIS IS THE LIDAR EQUATION. $P(r)$ IS MEASURED. σ AND β ARE TO BE DETERMINED.

WE DEFINE THE LOGARITHMIC RANGE-ADJUSTED POWER,

$$S(r) \equiv \ln [r^2 P(r)],$$

AND INTRODUCE A CONSTANT REFERENCE RANGE r_0 , SO THAT

$$S_0 \equiv S(r_0),$$

$$\beta_0 = \beta(r_0),$$

$$\sigma_0 = \sigma(r_0), \text{ AND SO ON.}$$

THE LIOR EQUATION MAY NOW BE NORMALIZED TO REMOVE SYSTEM-DEPENDENT ~~CONSTANT~~ FACTORS:

$$S(r) = S_0 + \ln \frac{\beta}{\beta_0} - 2 \int_{r_0}^r \sigma(r') dr'.$$

DIFFERENTIATING WRT r ,

$$\text{MEASURED} \rightarrow \frac{dS}{dr} = \frac{1}{\beta} \frac{d\beta}{dr} - 2\sigma$$

← UNKNOWN
← UNKNOWN

TO SOLVE THIS EQUATION, WE MUST ASSUME A "CONSTITUTIVE RELATION" BETWEEN β AND σ :

$$\beta \propto \sigma^k$$

KLETT NOTES EVIDENCE OF THIS FROM THE LITERATURE, WITH TYPICAL VALUES

$$\frac{2}{3} \leq k \leq 1.$$

DOES THIS IDEA MAKE PHYSICAL SENSE? WHY WOULD σ BE RELATED TO β ?

DISCUSS.

WE NOW HAVE

$$\frac{ds}{dr} = \frac{k}{\sigma} \frac{d\sigma}{dr} - 2\sigma,$$

A NONLINEAR ODE. WE COULD SOLVE NUMERICALLY, BUT THERE IS A 300 YEAR-OLD ANALYTIC SOLUTION (!)

~~WE KNOW~~

THE ABOVE IS A VERSION OF BERNOULLI'S EQUATION (SEE ANY GOOD ODE TEXT, OR RHB §14.2.7). WE MAKE THE SUBSTITUTION

$$\sigma = u^{-1}$$

$$\Rightarrow d\sigma = -u^{-2} du$$

$$\Rightarrow \frac{ds}{dr} = k u \left(-u^{-2} \frac{du}{dr} \right) - 2u^{-1}$$

$$\Rightarrow \frac{du}{dr} + \frac{u}{k} \frac{ds}{dr} = -\frac{2}{k} \quad *$$

WE NOW HAVE A LINEAR ODE, WHICH CAN BE SOLVED BY MEANS OF AN INTEGRATING FACTOR $\mu(x)$, CHOSEN ACCORDING TO THE REQUIREMENT:

$$\mu(r) \left[\frac{du}{dr} + \frac{u}{k} \frac{ds}{dr} \right] = \frac{d}{dr} \left[\mu(r) u \right] \quad \left(\text{SEE RHB §14.2.4} \right)$$

$$\Rightarrow \cancel{\mu} \frac{du}{dr} + \frac{\mu u}{k} \frac{ds}{dr} = u \frac{d\mu}{dr} + \mu \cancel{\frac{du}{dr}}$$

$$\Rightarrow ds = \frac{k d\mu}{\mu} \quad \text{FOR CONVENIENCE,}$$

$$s = k \ln \mu + C, \quad \mu = e^{(s-s_0)/k}$$

FOR CONVENIENCE, CHOOSE $C = S_0$.

SO WE MULTIPLY THIS BY $*$ TO OBTAIN

$$\frac{d}{dr} \left[u e^{(s-s_0)/k} \right] = -\frac{2}{k} e^{(s-s_0)/k}$$

$$\Rightarrow u e^{(s-s_0)/k} \Big|_{r_0}^r = -\frac{2}{k} \int_{r_0}^r e^{(s-s_0)/k} dr'$$

$$\Rightarrow u(r) e^{(s-s_0)/h} - u(r_0) = -\frac{2}{h} \int_{r_0}^r e^{(s(r')-s_0)/h} dr'$$

$$u(r) = \frac{u(r_0) - \frac{2}{h} \int_{r_0}^r e^{(s(r')-s_0)/h} dr'}{e^{(s-s_0)/h}}$$

$$\therefore \sigma(r) = \frac{e^{(s-s_0)/h}}{\left[\sigma_0^{-1} - \frac{2}{h} \int_{r_0}^r e^{(s(r')-s_0)/h} dr' \right]}$$

TENDS TO A 0/0 LIMIT AS r INCREASES

KLETT POINTS OUT THAT THIS SOLUTION IS UNSTABLE. AS r INCREASES, THE SOLUTION TENDS TO DIVERGE.

HE PROPOSES THE ALTERNATIVE

$$\sigma(r) = \frac{e^{(s-s_m)/h}}{\left[\sigma_m^{-1} + \frac{2}{h} \int_r^{r_m} e^{(s(r')-s_m)/h} dr' \right]}$$

STABLE AS r INCREASES

WHICH IS IDENTICAL — BUT THE INTENT IS THAT THE NEW REFERENCE RANGE, r_m , IS AT THE MAXIMUM RANGE THAT GIVES US USABLE SIGNAL.

AS A RESULT,

- $\sigma(r)$ DOES NOT DIVERGE
- $\sigma(r)$ IS RELATIVELY INSENSITIVE TO σ_m , WHICH MAY BE DIFFICULT TO ESTIMATE.