

Stadium Billiards

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Contents

1	Introduction	1
2	The Bunimovich Stadium	1
3	Classical Stadium Billiards	2
3.1	Ray Tracing	2
3.2	Stability of Periodic Orbits	2
3.3	Lyapunov Exponent	2
3.4	Ergodicity	2
4	Quantum Eigenfunctions of the Stadium	2
4.1	Demonstration	3
4.2	Heller's PWDM Solution	3
4.3	Scarring	4
4.4	Comments	4

1 Introduction

We now consider the classical and quantum dynamics of a point particle on a flat, 2D “table,” with a perfectly reflecting boundary. Specifically, we will focus on the Bunimovich Stadium. This deceptively simple system is interesting because:

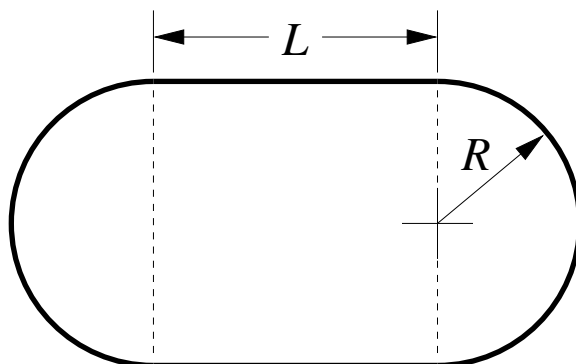
1. Classically, it exhibits chaotic behavior.

2. Quantum mechanically, its eigenmode structure is rich and surprising.
3. The transition between the classical and quantum behaviors is curious and, to a some extent, mysterious.

Moreover, both the classical and quantum billiard problems illustrate an important lesson about scientific computation: *Generic methods can only take you so far*. Whether we are constructing a model or analyzing data, the most fruitful approaches are formulated with the specific problem squarely in mind.

2 The Bunimovich Stadium

Our billiards table is the Bunimovich Stadium (below). It is a rectangle with circular end caps. Most of my examples use $R = 1$ and $L = 1$.



3 Classical Stadium Billiards

3.1 Ray Tracing

3.2 Stability of Periodic Orbits

3.3 Lyapunov Exponent

3.4 Ergodicity

4 Quantum Eigenfunctions of the Stadium

In units where $\hbar = 2m = 1$, the time-independent Schrödinger equation is

$$[-\nabla^2 + V(\mathbf{r})] \psi(\mathbf{r}) = E \psi(\mathbf{r}). \quad (1)$$

The potential $V(\mathbf{r})$ is zero inside the stadium, and infinite outside. So, this is a 2D variation on the particle in a box. Before we embark on this problem, let me remark on the physical significance of the solutions.

Of course, in an infinite well there are no free particle states. Any initial condition can be expressed as a linear combination of the bound eigenfunctions:

$$\Psi(\mathbf{r}, 0) = \sum_{n=1}^{\infty} c_n \psi_n(\mathbf{r}). \quad (2)$$

The subsequent time evolution is given by

$$\Psi(\mathbf{r}, t) = \sum_{n=1}^{\infty} c_n \psi_n(\mathbf{r}) e^{-iE_n t}. \quad (3)$$

In other words, the future evolution of the system is fully, one might even say trivially, determined by the initial conditions. This is true even in the classical (large n) limit. Let that sink in. Does this result contradict what we know about the chaotic dynamics of the classical system?

4.1 Demonstration

Since the solutions to the Schrödinger equation on a flat potential are just plane waves, the normal modes of the quantum mechanical stadium are identical those for the wave equation with homogeneous Dirichlet boundary conditions. They can be discovered experimentally in an almost infinite number

of setups, including acoustic waves, microwaves, or waves on an elastic membrane.

Jerry DiMarco has built an elegant demonstration that uses a function generator and a loop of wire with a soap film to find the normal modes of the stadium. See Arcos et al., AJP 1998 **66**, 601.

Q: How does the time dependence of quantum eigenfunctions differ from that of the scalar wave equation modes?

4.2 Heller’s PWDM Solution

E. J. Heller (1987 *Phys. Rev. Lett.* **53**: 1515) invented the plane wave decomposition method (PWDM) to solve the Schrödinger equation on a flat billiard of any shape. This presentation and my example codes closely follow Li, Robnik and Hu (1988 *Phys. Rev. E* **57**, 4095).

We seek energy eigenfunctions $\psi(\mathbf{r})$ for $\mathbf{r} \in \mathcal{B}$, with potential $V(\mathbf{r}) = 0$ subject to $\psi(\mathbf{r}) = 0$ on $\partial\mathcal{B}$. We approximate the eigenfunction using a sum of N plane waves,

$$\psi(\mathbf{r}) = \sum_{j=1}^N a_j \cos(k(x \cos \theta_j + y \sin \theta_j) + \phi_j). \quad (4)$$

The phases ϕ_j are random, and the directions are $\theta_j = 2\pi j/N$. This *ansatz* (4) satisfies Schrödinger’s equation.

Choose M “primary nodes,” $\mathbf{p}_n \in \partial\mathcal{B}$. A final point, \mathbf{p}_{M+1} , is chosen in the interior. Our Dirichlet boundary condition can be written:

$$\psi(\mathbf{p}_n) = \begin{cases} 0 & \text{for } n \leq M, \\ 1 & \text{for } n = M + 1. \end{cases} \quad (5)$$

Note that the interior point is being used to avoid the trivial result, $a_j = 0$. Since only the coefficients a_j are unknown, we recast (5) as a matrix equation:

$$\mathcal{M} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{M+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad (6)$$

The matrix \mathcal{M} will generally be ill-conditioned, so we invert \mathcal{M} using *singular value decomposition*.¹

How do we know whether k , or equivalently $E = k^2$, is an eigenvalue? First, we adjust our a_j in concert to normalize ψ since the interior point choice was quite arbitrary. Now choose mM “secondary nodes,” \mathbf{q}_n , between the primary nodes, where m is some integer. We define the tension,

$$T(k) = \frac{1}{mM} \sum_{n=1}^{mM} \psi^2(\mathbf{q}_n). \quad (7)$$

The eigenvalues occur at minima of T . Conventional choices are:

$$N = \frac{bL}{\lambda}, \quad b = 12, \quad M = \frac{5N}{3}, \quad m = 3, \quad (8)$$

where L is the perimeter of the billiard table and $\lambda = 2\pi/k$ is the *de Broglie* wavelength.

4.3 Scarring

4.4 Comments

- What range of problems can be solved by PWDM?
- Many eigenfunctions exhibit “scarring,” regions of enhanced amplitude that resemble the classically unstable periodic orbits. Apparently in QM they are stable, periodic solutions! Are they exact solutions, and do they exist for arbitrarily large n ?
- Describe the symmetry of stadium eigenfunctions.
- Some of the modes found by PWDM do not display the expected symmetry. Why? Can all the modes affected by this problem can be discovered by testing for symmetry?
- It would be interesting to construct a wave packet by superposition of eigenfunctions. The exact evolution of the wave packet could then be determined using equation 3. How would you “clean up” the PWDM results for this task?

¹*Numerical Recipes* §2.6.