

The Applied Optics of *MOSES*

Lectures for ECE 580

Charles C. Kankelborg
Department of Physics
Montana State University
Bozeman, MT 59717
kankel@mithra.physics.montana.edu

September 24 & 26, 2002

Abstract

The *Multi-Order Solar EUV Spectrograph (MOSES)* is a new type of slitless imaging spectrograph with detectors at three spectral orders ($n = -1, 0, 1$). This presentation summarizes the basic optical physics behind the instrument and introduces the “ill-posed inversion problem” as it relates to *MOSES* data.

The MOSES project is funded by NASA Grant NAG5-10997.

Outline

1. Propagation in a dissipative medium
2. Reflection at an interface
3. Multilayer coatings (“layered synthetic microstructure” , LSM)
4. Imaging spectroscopy
5. Inverse problem for *MOSES*

Dissipative Media and Multilayer Mirrors

We have been studying optics under the following assumptions:

- Propagation without dissipation
- Rayleigh-Sommerfeld diffraction
 - propagation strictly from left to right
 - thin screen
 - structures with length scale $L \gg \lambda$.

We now lay aside these assumptions to consider propagation in a dissipative medium. In EUV light, all media (other than high vacuum) are extremely dissipative.

Electrodynamics in a Conductor *

Maxwell's equations in a linear medium:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\epsilon, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \mu \mathbf{J} + \mu \epsilon \frac{\partial \mathbf{E}}{\partial t},\end{aligned}\tag{1}$$

In a conductor, $\mathbf{J} = \sigma \mathbf{E}$ (Ohm's law). Conservation of charge:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot \mathbf{J} \\ &= -\sigma (\nabla \cdot \mathbf{E}) \\ &= -\frac{\sigma}{\epsilon} \rho \\ \therefore \rho(t) &= \rho(0) e^{-(\sigma/\epsilon)t}\end{aligned}$$

For large times $t \gg \sigma/\epsilon$, $\rho \rightarrow 0$.

*David J. Griffiths, *Introduction to Electrodynamics*, Prentice-Hall, 1981

Maxwell's equations in a conductor are therefore:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t},\end{aligned}\tag{2}$$

The wave equation is now

$$\nabla^2 \mathbf{E} = \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{E}}{\partial t}.\tag{3}$$

As in vacuum, the free space solutions are transverse plane waves such as

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{j}} e^{i(\kappa x - \omega t)}.\tag{4}$$

The wavenumber κ is now complex, and is expressed in terms of a complex index of refraction N :

$$\kappa = \pm \frac{\omega}{c} N, \quad N \equiv n + ik \in \mathbb{C}.\tag{5}$$

The \pm indicates waves traveling right or left.

Exercise 1 *Verify the derivation of the wave equation (3) from Maxwell's equations (2). Show that it is the same for $\mathbf{B}(x, t)$.*

Exercise 2 *Verify the plane wave solution (4). Show that the same form is possible for $\mathbf{B}(x, t)$. Looking back at Maxwell's equations (2), show that only transverse waves are possible. **

*Do not be fooled, however: these waves are more interesting than EM waves in vacuum. As Griffiths points out, the electric and magnetic fields are out of phase by an amount that depends on the material properties!

Dimensionless **optical constants** n and k apply more generally than the foregoing derivation, but have the following form in a conductor:

$$\text{refractive index } n(\omega) = c \sqrt{\frac{\epsilon\mu}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]}, \quad (6)$$

$$\text{extinction coefficient } k(\omega) = c \sqrt{\frac{\epsilon\mu}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]}. \quad (7)$$

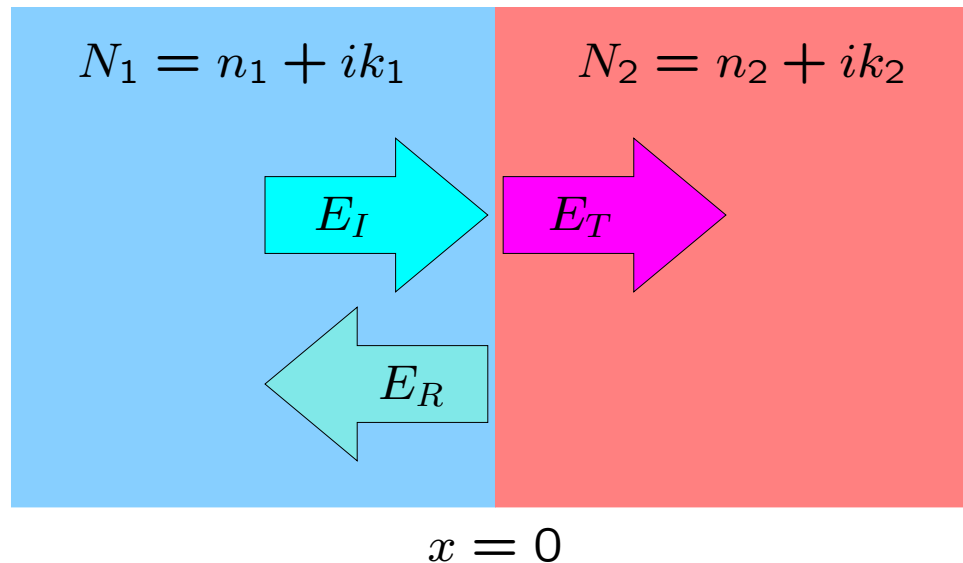
N.B.: The conductivity $\sigma(\omega)$, permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$ are all functions of frequency!

Equation 4 for a plane-wave may be rewritten for a material j with optical constants n_j, k_j :

$$E(x, t) = E_0 e^{-(\omega/c)k_j x} e^{i(\omega/c)n_j x} e^{-i\omega t}. \quad (8)$$

Reflection and Transmission at an Interface

A transverse plane wave traveling in the $+x$ direction through medium 1 is normally incident on an interface with medium 2 at $x = 0$.



We assume a solution that includes complex amplitudes for the incident wave E_I , a reflected wave E_R , and a transmitted wave E_T . Dividing out the time harmonic factor $e^{-i\omega t}$, the total field is

$$E(x) = \begin{cases} E_I e^{i(\omega/c)N_1x} + E_R e^{-i(\omega/c)N_1x}, & x \leq 0, \\ E_T e^{i(\omega/c)N_2x}, & x > 0. \end{cases} \quad (9)$$

The boundary conditions at $x = 0$ are continuity of $E(x)$ and of $E'(x)$. From this, we may derive the **Fresnel coefficients** for the reflected and transmitted waves,

$$\mathcal{F}_R \equiv \frac{E_R}{E_I} = \frac{N_1 - N_2}{N_1 + N_2} \in \mathbb{C}, \quad \mathcal{F}_T \equiv \frac{E_T}{E_I} = \frac{2N_1}{N_1 + N_2} \in \mathbb{C}. \quad (10)$$

Then, the power reflection and transmission coefficients are

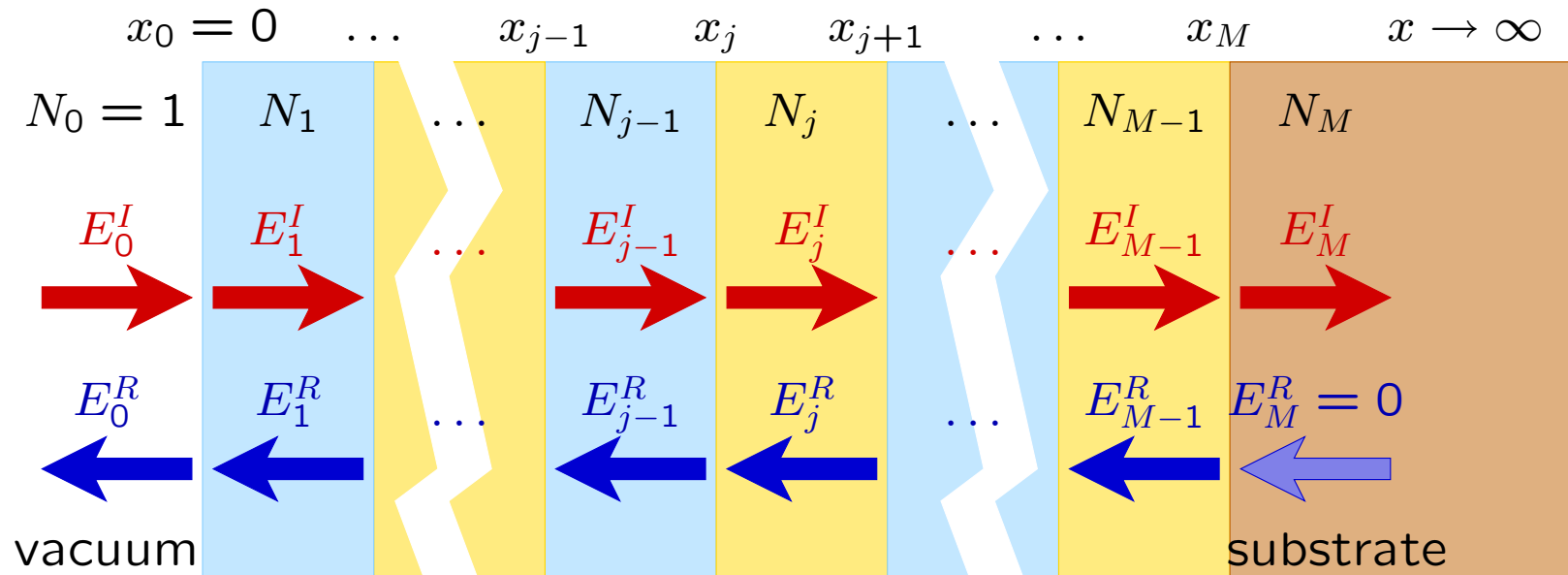
$$R = \frac{E_R^* E_R}{E_I^* E_I} = \mathcal{F}_R^* \mathcal{F}_R, \quad T = \frac{E_T^* E_T}{E_I^* E_I} = \mathcal{F}_T^* \mathcal{F}_T. \quad (11)$$

The most reflective substances in EUV are elements with high atomic number Z . For example, the reflection coefficient of Iridium ($Z = 77$) is about 3% at $\lambda 304 \text{ \AA}$ (41 eV), the operating wavelength for *MOSES*.

Exercise 3 Consider the solution (eq. 9) in light of Maxwell's equations. What would be the consequences of a discontinuity at the interface $x = 0$ in **(a)** the transverse electric field $E(x)$? **(b)** in $E'(x)$?

Exercise 4 Verify that the Fresnel coefficients (eq. 10) satisfy the boundary conditions.

Multilayer Coatings *



Now assume $E_M^R = 0$ (substrate doesn't backscatter photons into the multilayer). Seek recursion relations that will allow us to work right to left.

*J.H.Underwood and T.W.Barbee, Jr. *Appl.Opt.* **20**, 3027

Continuity of $E(x)$ at $x = x_j$:

$$\begin{aligned} E_{j-1}^I e^{i(\omega/c)N_{j-1}x_j} + E_{j-1}^R e^{-i(\omega/c)N_{j-1}x_j} \\ = E_j^I e^{i(\omega/c)N_jx_j} + E_j^R e^{-i(\omega/c)N_jx_j} \end{aligned} \quad (12)$$

Continuity of $E'(x)$ at $x = x_j$:

$$\begin{aligned} N_{j-1}E_{j-1}^I e^{i(\omega/c)N_{j-1}x_j} - N_{j-1}E_{j-1}^R e^{-i(\omega/c)N_{j-1}x_j} \\ = N_jE_j^I e^{i(\omega/c)N_jx_j} - N_jE_j^R e^{-i(\omega/c)N_jx_j} \end{aligned} \quad (13)$$

Take (eq. 12) + $\frac{1}{N_{j-1}}$ (eq. 13) and solve for E_{j-1}^I :

$$E_{j-1}^I = \frac{1}{2} \left(1 + \frac{N_j}{N_{j-1}} \right) E_j^I e^{i(\omega/c)(N_j - N_{j-1})x_j} + \frac{1}{2} \left(1 - \frac{N_j}{N_{j-1}} \right) E_j^R e^{-i(\omega/c)(N_j + N_{j-1})x_j} \quad (14)$$

Take (eq. 12) - $\frac{1}{N_{j-1}}$ (eq. 13) and solve for E_{j-1}^R :

$$E_{j-1}^R = \frac{1}{2} \left(1 - \frac{N_j}{N_{j-1}} \right) E_j^I e^{i(\omega/c)(N_j + N_{j-1})x_j} + \frac{1}{2} \left(1 + \frac{N_j}{N_{j-1}} \right) E_j^R e^{-i(\omega/c)(N_j - N_{j-1})x_j} \quad (15)$$

For every frequency ω of interest, we follow this solution procedure to obtain reflectivity and phase shift:

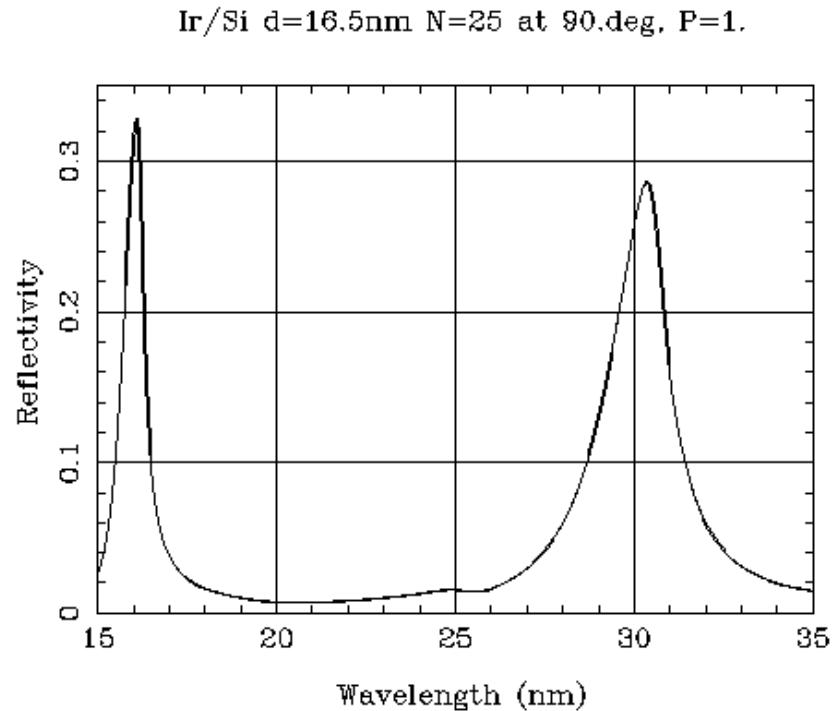
1. Let $E_M^R = 0$ and $E_M^I = 1$.
2. Use the recursion relations (eqs. 14,15) to calculate E_0^R and E_0^I .
3. The Fresnel coefficient for the multilayer mirror is

$$\mathbb{F}_R = \frac{E_0^R}{E_0^I}. \quad (16)$$

4. The reflectivity is

$$R = |\mathbb{F}_R|^2. \quad (17)$$

Our solution procedure will work for an arbitrarily complex LSM. It can be modified to treat non-normal incidence and polarization. There is a web site with optical constants, as well as a cgi script for modeling simple (periodic) multilayers. An example:



http://www-cxro.lbl.gov/optical_constants/

Spectrometers

Diffraction Grating

Consider a plane wave incident at angle α on a screen of parallel slits with regular spacing d . The transmitted intensity is maximized in the far field for angles β that satisfy the **grating equation**:

$$m\lambda = d(\sin \alpha + \sin \beta), \quad m \in \mathbb{I} \quad (18)$$

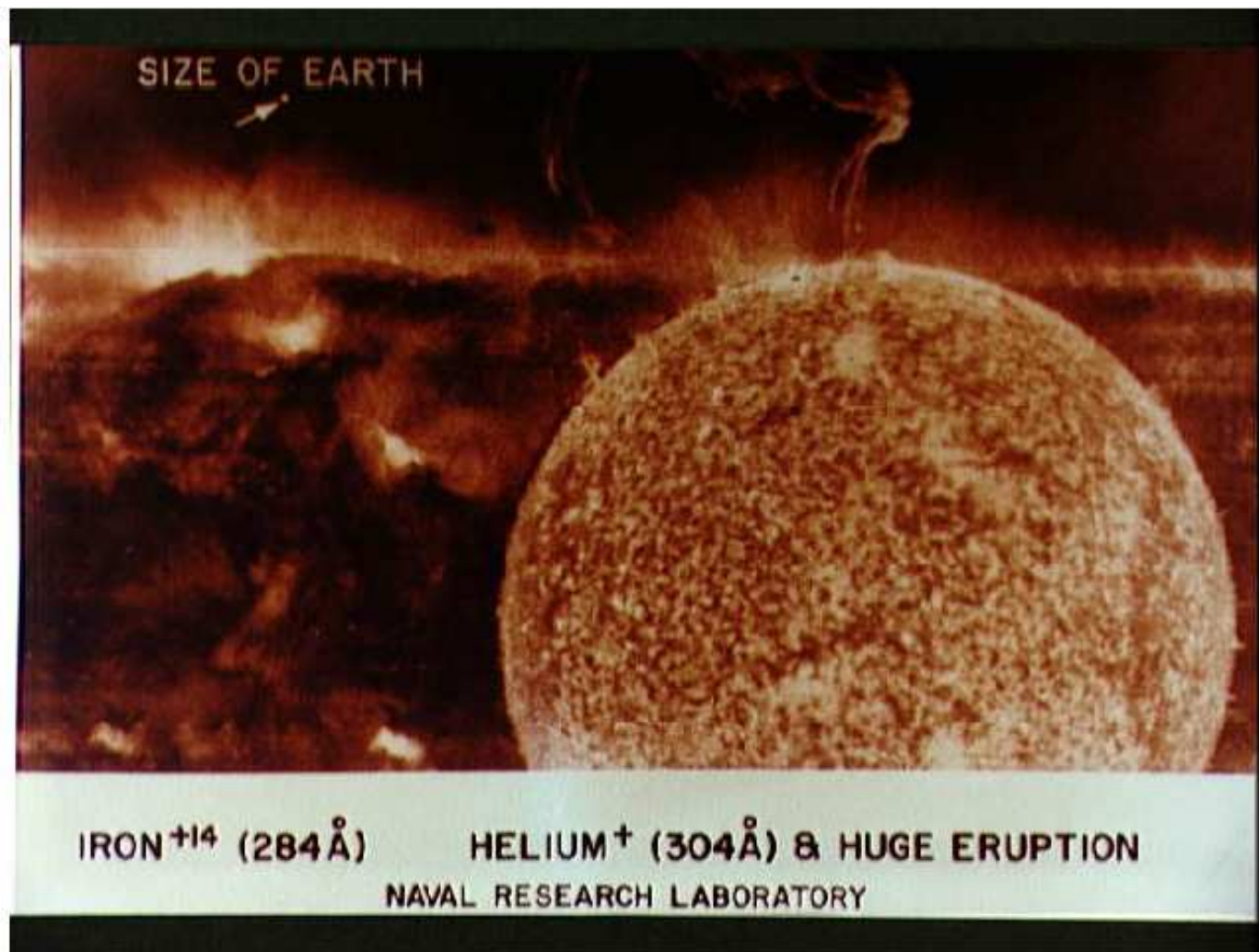
Exercise 5 *What is the Fourier transform of a square wave with period d ? What does this have to do with a diffraction grating?*

A Slitless Spectrograph

- An objective grating may be used form overlapping images in several emission lines.
- Because the spectral and spatial information are convolved, interpretation is difficult and subjective.
- In spite of this difficulty, many good science results were obtained using data from the Skylab S082A spectroheliograph.



Result: intensity $I(x + nD\lambda, y)$.



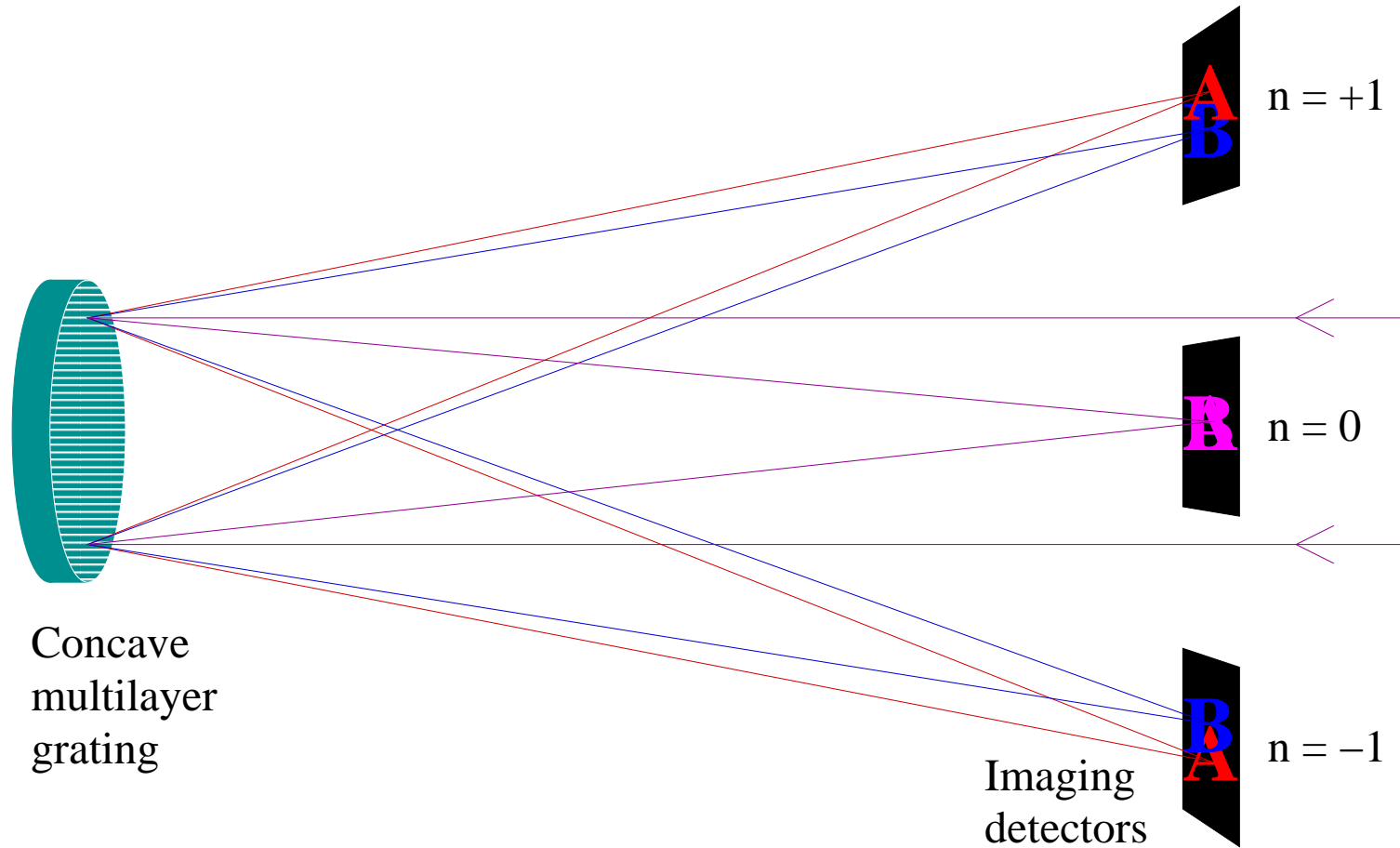
Part of an “overlappograph” image taken by the S082A EUV spectroheliograph on the Skylab Apollo Telescope Mount, January 1, 1974 (*JSC Digital Image Collection*)

Slit Spectrograph

- Telescope produces light in focal plane f
- Slit at f picks off a narrow range in x
- Concave grating re-images slit onto detector with dispersion.

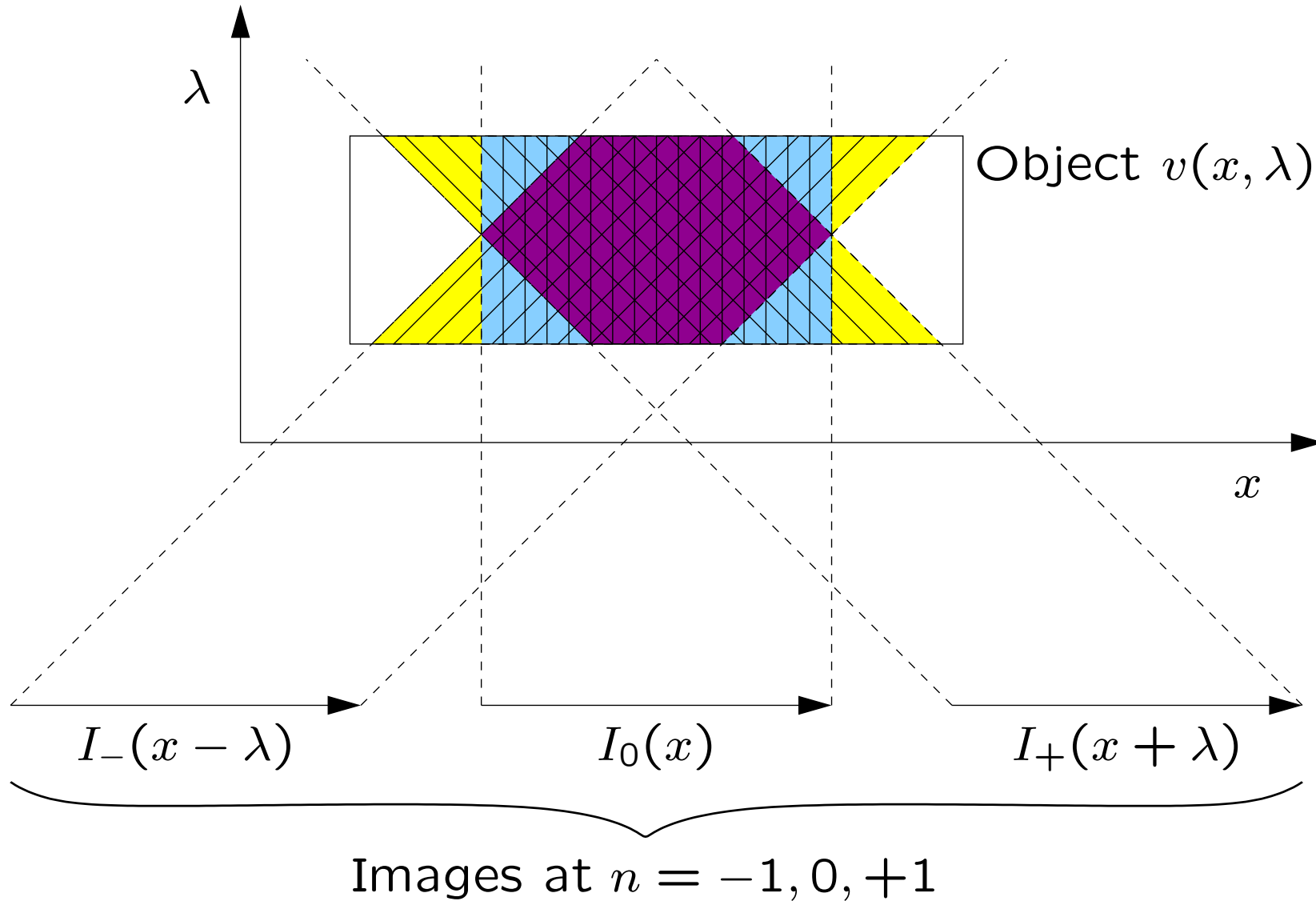
Result: intensity $I(\lambda, y)$.

A 3-Order Slitless Spectrograph



Result: $I(x + D\lambda)$, $I(x)$, $(x - D\lambda)$.

Projection through (x, λ) space



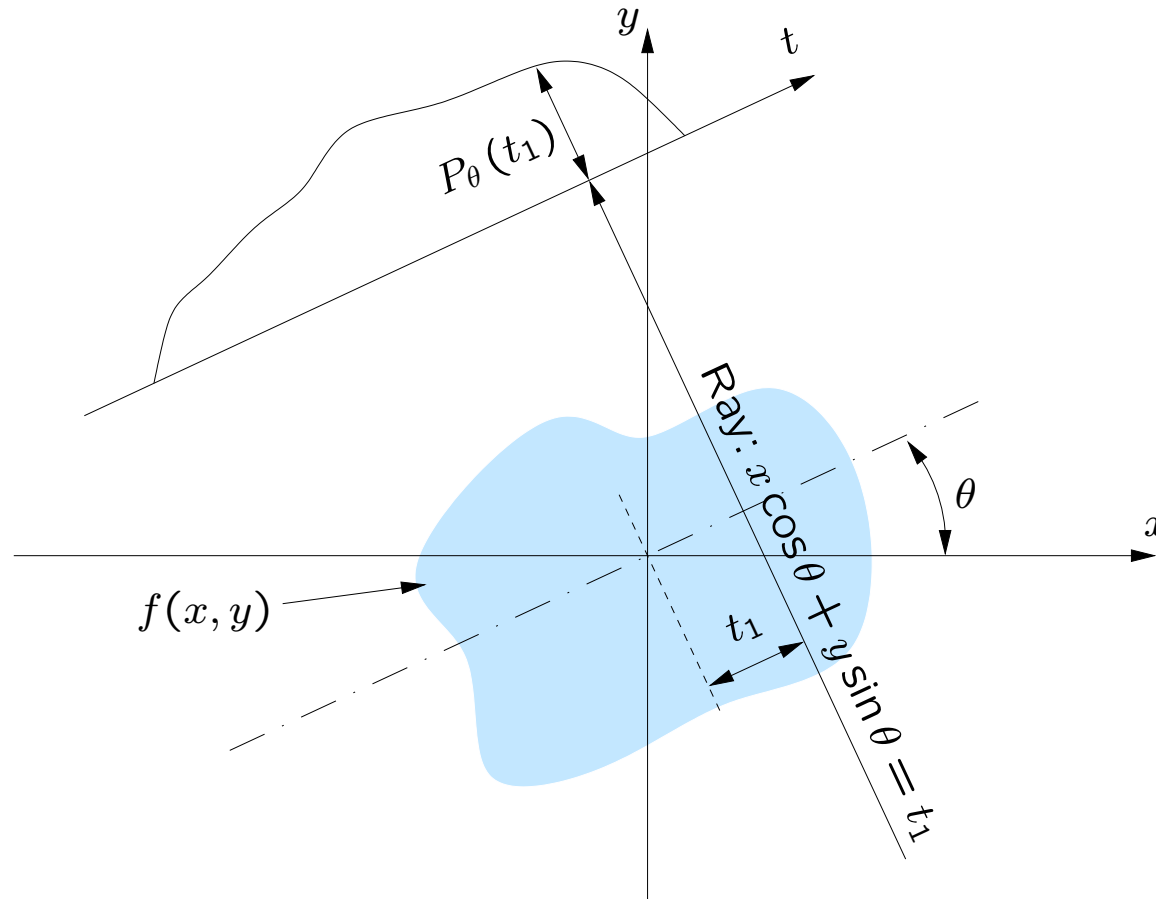
The Big Question: Inversion

Can we turn $I_+(x + \lambda)$, $I_0(x)$, $I_-(x - \lambda)$ back into $I(x, \lambda)$? And if so, how?

The inversion problem seems poorly constrained: solve for $M \times N$ parameters from $M \times 3$ data points!

Exercise 6 *Solve the inverse problem for MOSES. (Just kidding!)*

The Radon Transform *



*Material in the next several slides relies heavily on Chapter 3 of A.C. Kak and Malcolm Slaney, *Principles of Computerized Tomographic Imaging*, IEEE Press, 1988. Available free: <http://www.slaney.org/pct/>

The Radon transform is defined by

$$\begin{aligned}
 P_\theta(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - t) dx dy \\
 &= \int_{-\infty}^{\infty} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds.
 \end{aligned} \tag{19}$$

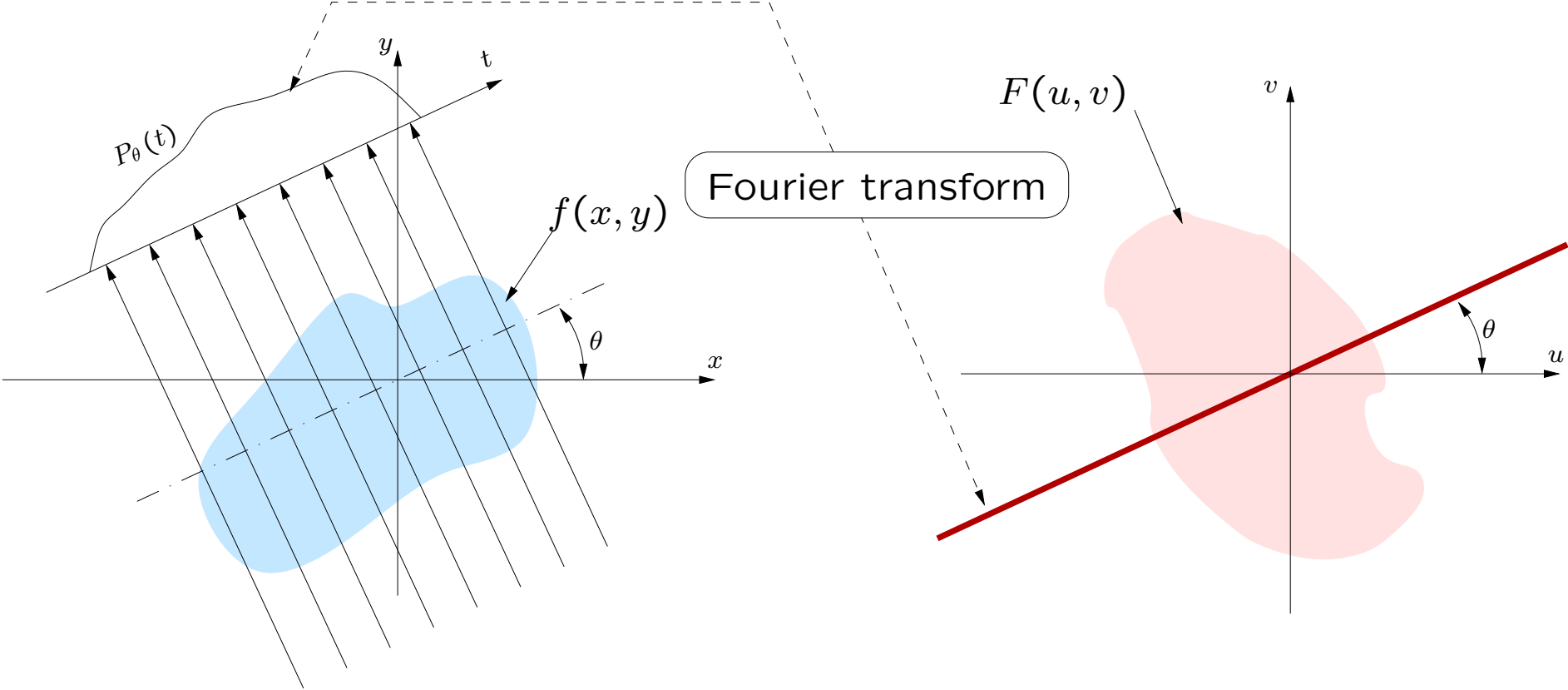
The Fourier transform $S_\theta(w)$ of $P_\theta(t)$ corresponds to a slice of $F(u, v)$, the 2D Fourier transform of $f(x, y)$:

$$S_\theta(w) = F(w \cos \theta, w \sin \theta). \tag{20}$$

Exercise 7 *Verify equation 20. You may wish to use the fact that the 2D Fourier transform merely rotates if the function is rotated. That is,*

$$\begin{array}{l}
 \text{if} \\
 \text{then}
 \end{array}
 \begin{array}{l}
 F \{f(r, \theta)\} = F(\omega, \phi) \\
 F \{f(r, \theta + \alpha)\} = F(\omega, \phi + \alpha).
 \end{array} \tag{21}$$

The **Fourier Slice Theorem** (eq. 20) is the fundamental theorem of tomography.



MOSES Backprojection Algorithm

1. Create a blank image $F(u, v)$ that will hold the FFT of the reconstructed image $I(x, \lambda)$.
2. Take the Fourier Transform $S_n(k)$ of each of the three projections I_+, I_-, I_0 .
3. Use the F_n to fill in the appropriate slices of $F(u, v)$.
4. Apply the inverse transform to obtain an initial guess $I(x, \lambda)$.

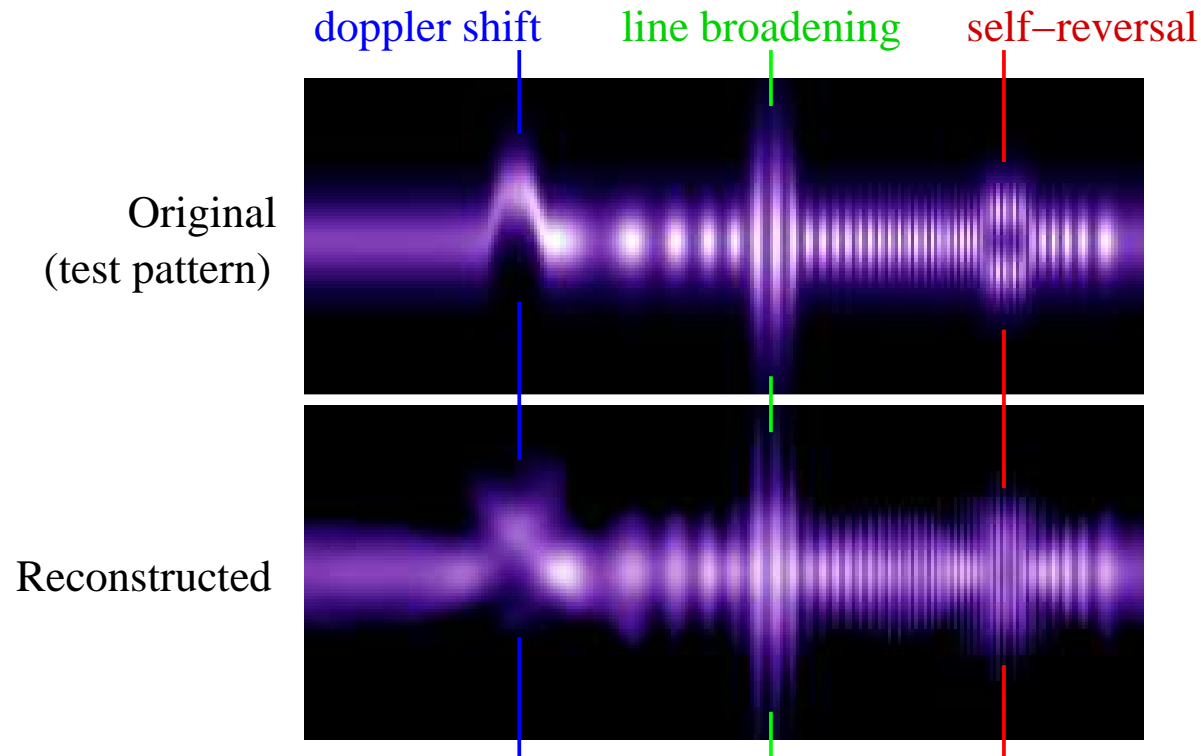
In practice, it helps to know (or be able to guess) the projection along x , $\int_{-\infty}^{\infty} I(x, \lambda) dx$.

Problem: lots of negative counts in the backprojected image.

We may use an iterative approach to enforce non-negativity:

1. Zero all negative elements of the backprojection.
2. Fourier transform the result, use it to fill the wedges of $F(u, v)$ that are not constrained by the data.
3. Inverse Fourier transform to obtain an improved backprojection.
4. Repeat until the sum of the negative counts in the backprojection is small compared to the total uncertainty in image counts.
5. Zero all negative pixels in the backprojection.

Data Inversion Example



A test pattern is projected onto the three spectrograph orders and then reconstructed by Fourier backprojection with two added constraints: (1) the slit averaged spectrum and (2) non-negativity. The doppler shift and line width are preserved but the self-reversal is suppressed.