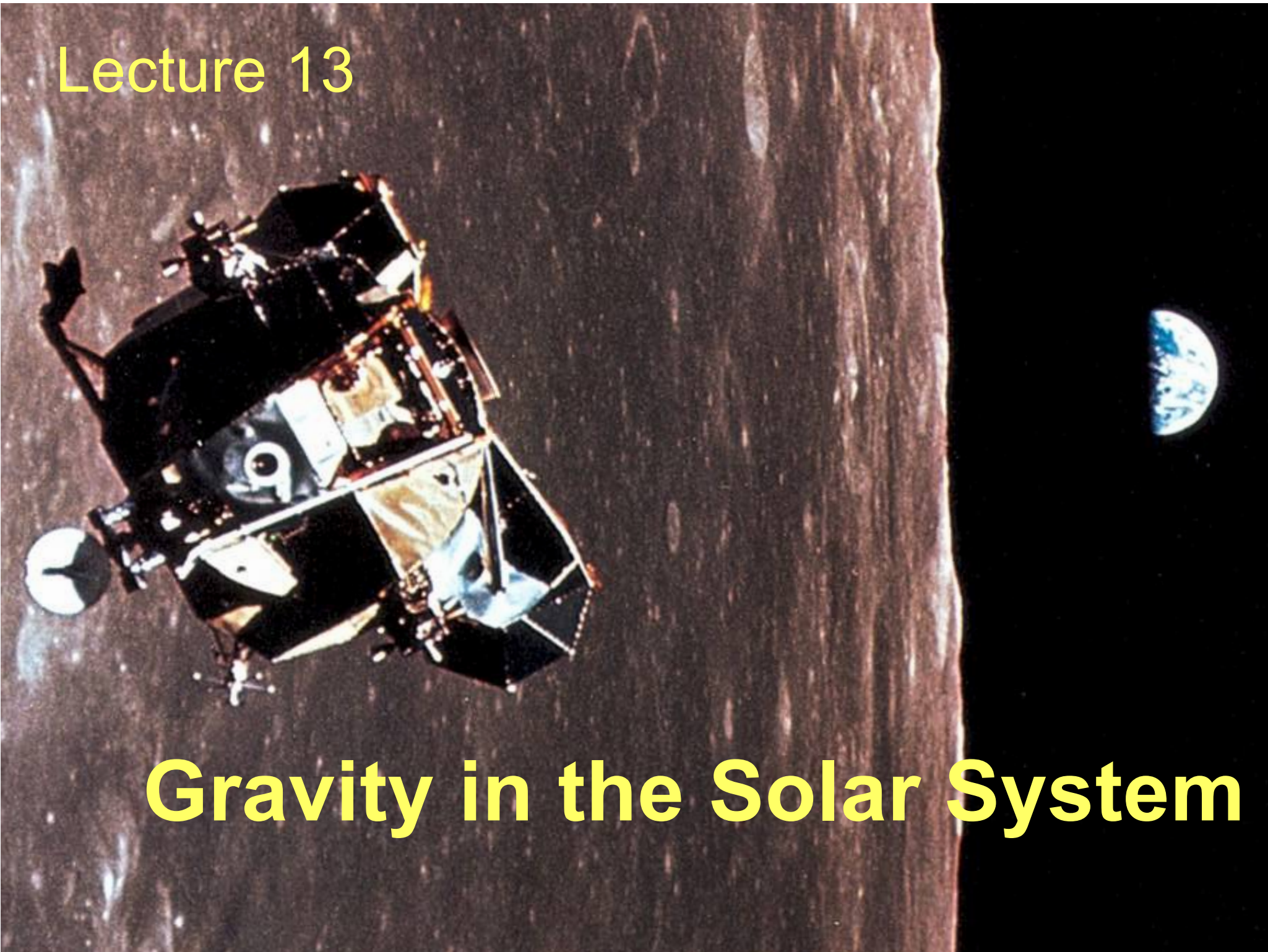


Lecture 13

Gravity in the Solar System



Guiding Questions

1. How was the heliocentric model established? What are monumental steps in the history of the heliocentric model?
2. How do Kepler's three laws describe the motions of planets in the solar system?
3. How does Newton's law of universal gravitation explain the motions of the planets?

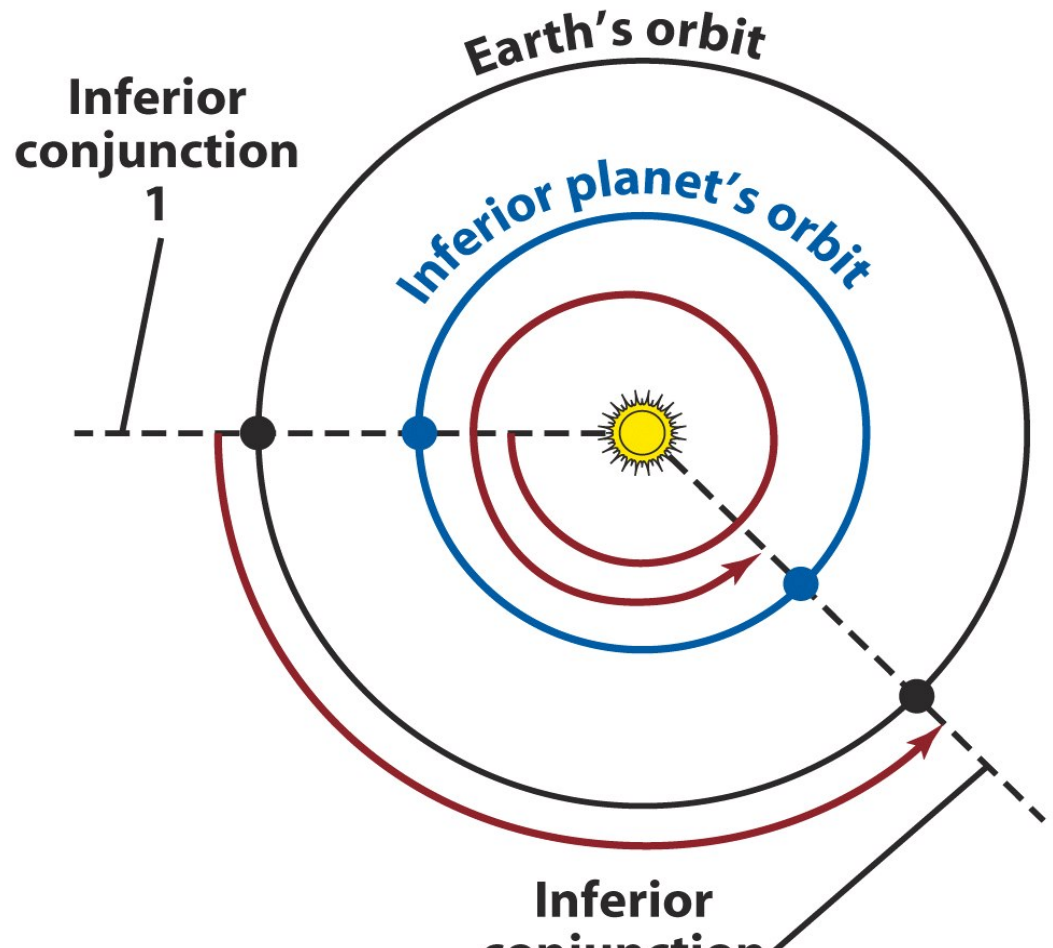
13.1 Overview: from Copernicus to Newton

- The motions of the planets in the solar system provide one of the best examples of how physical laws to understand nature develop from observations.
- Copernicus first devised the framework of **heliocentric** model to explain the motions of the planets.
- Tyco's invaluable precise measurements were corner stones for Kepler and Newton's laws.
- Kepler established three laws to describe motions in the solar system
- Newton's law of universal gravitation

Copernicus used a geometric method to determine the **distances** of the planets relative to the Sun-Earth distance, i.e., in units of AU. He also figured out the **synodic period** from observations, and calculated the **sidereal period** of each planet.

The **sidereal period** (P) of a planet, its true orbital period, is measured with respect to the stars.

Its **synodic period** (S) is measured with respect to the Earth and the Sun (for example, from one conjunction to the next)

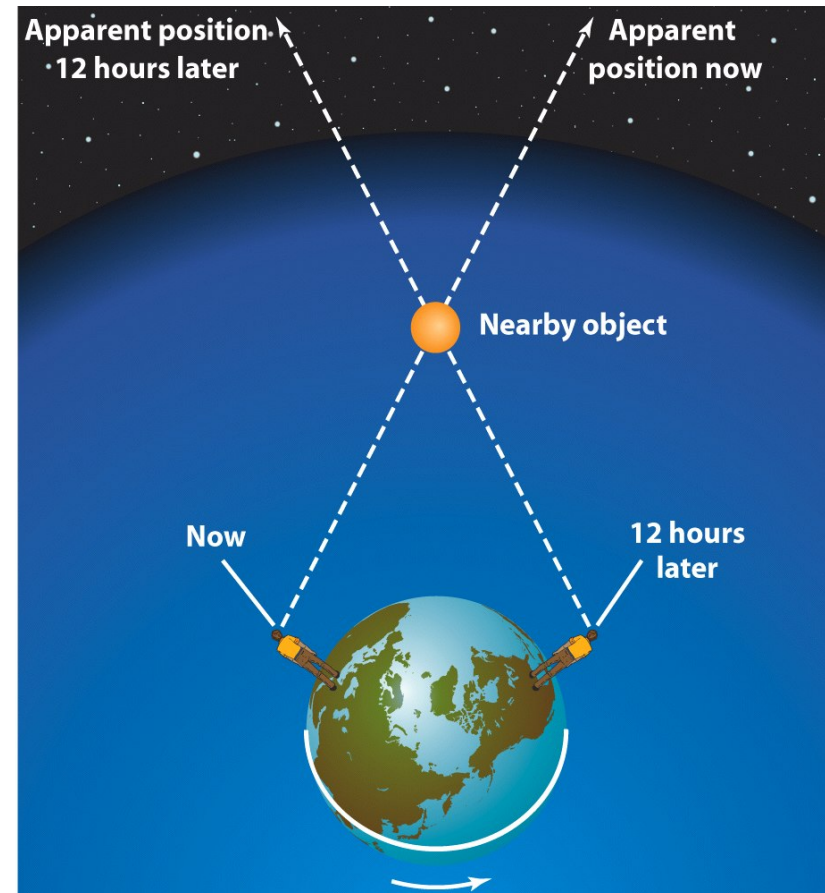


A planet further from the Sun has a longer sidereal period.

Tycho Brahe's astronomical observations disproved ancient ideas about the heavens



Tycho Brahe observing



Tycho tried to find the distance of stars by observing **parallax** shifts. His comprehensive measurements of positions of planets put the helio-centric model on a solid foundation.

13.2 Kepler's Three Laws

Johannes Kepler proposed **elliptical** paths for the planets about the Sun.

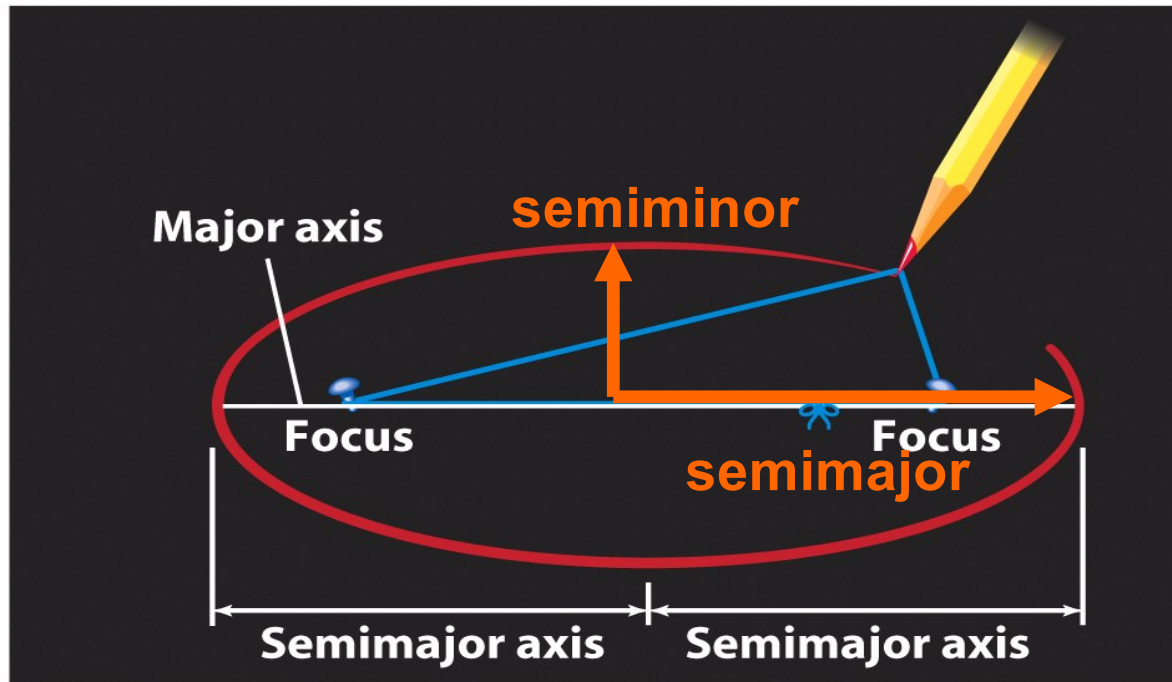


Using data collected by Tycho Brahe, Kepler deduced **three laws of planetary motion**:

1. the orbits are ellipses
2. a planet's speed varies as it moves around its elliptical orbit
3. the orbital period of a planet is related to the size of its orbit

Kepler's laws are a landmark in the history of astronomy. They are not only useful to understand planetary orbits, but are applied to celestial objects outside the solar system.

Kepler's First Law: the orbit of a planet is an ellipse with the Sun at **one focus**.



$$e = \frac{(R_a - R_p)}{(R_a + R_p)} = \frac{(R_a - R_p)}{2a}$$

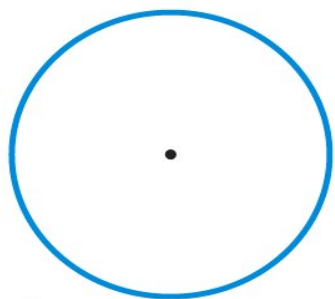
R_a = distance at Aphelion

R_p = distance at Perihelion

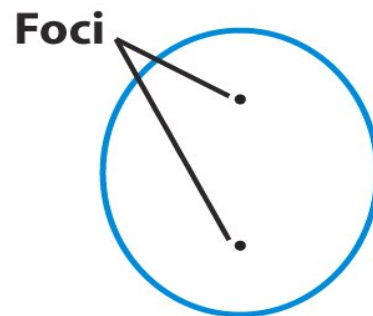
for a circle :

$$R_a = R_p, e = 0$$

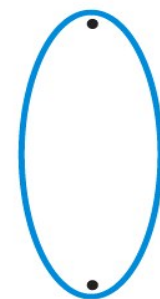
(a)



(b) $e = 0$



$e = 0.50$



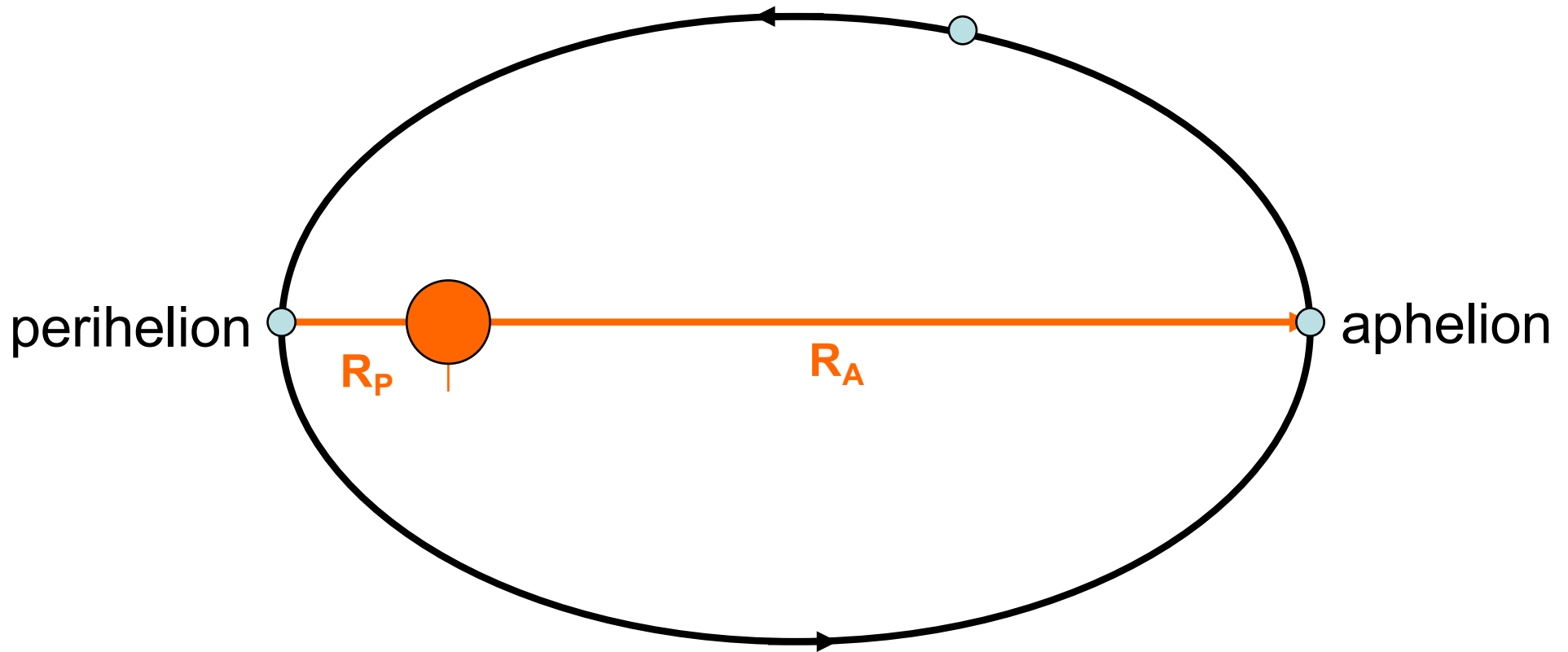
$e = 0.90$



$e = 0.99$

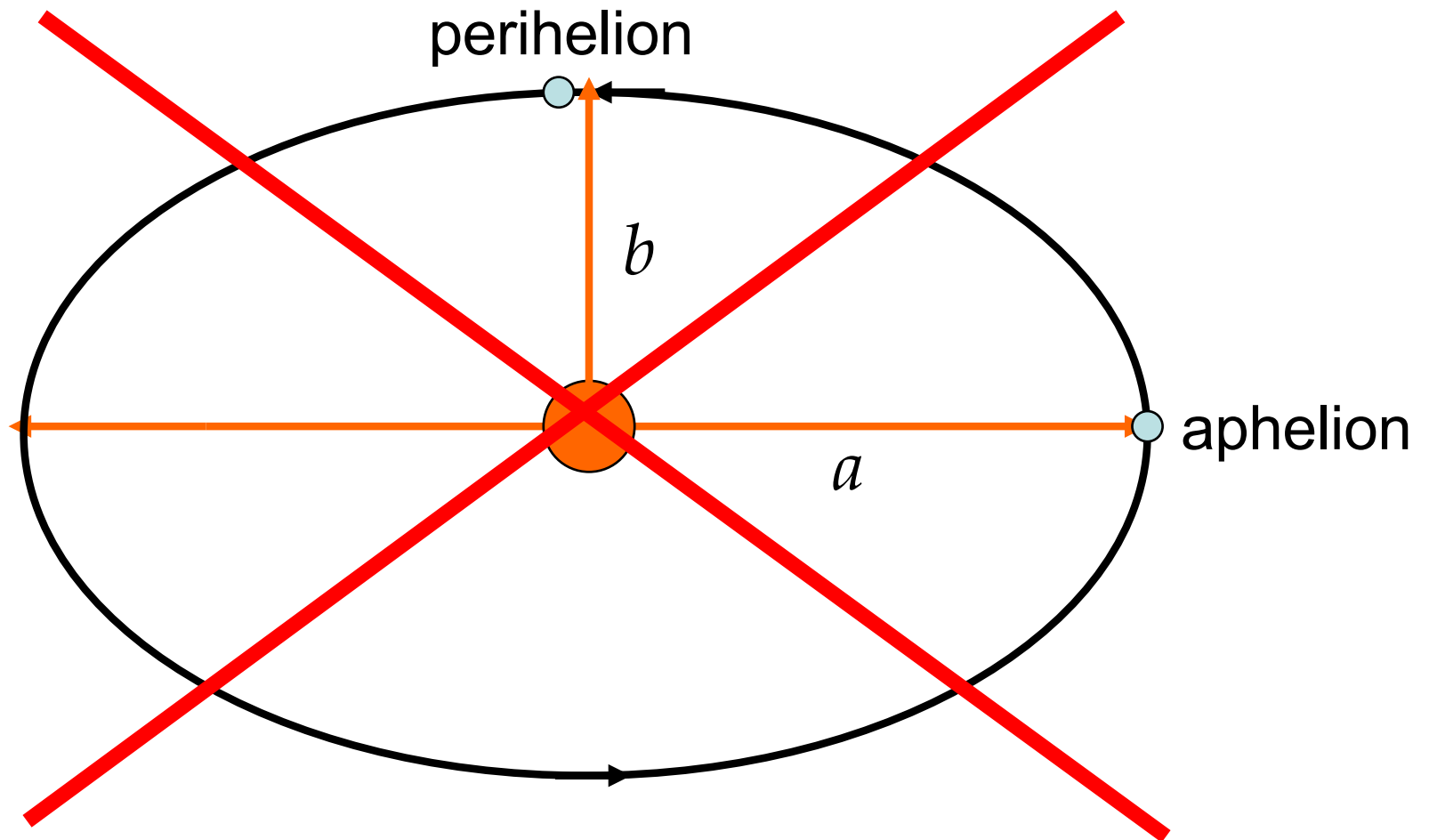
e : eccentricity

$$R_P + R_A = 2a$$



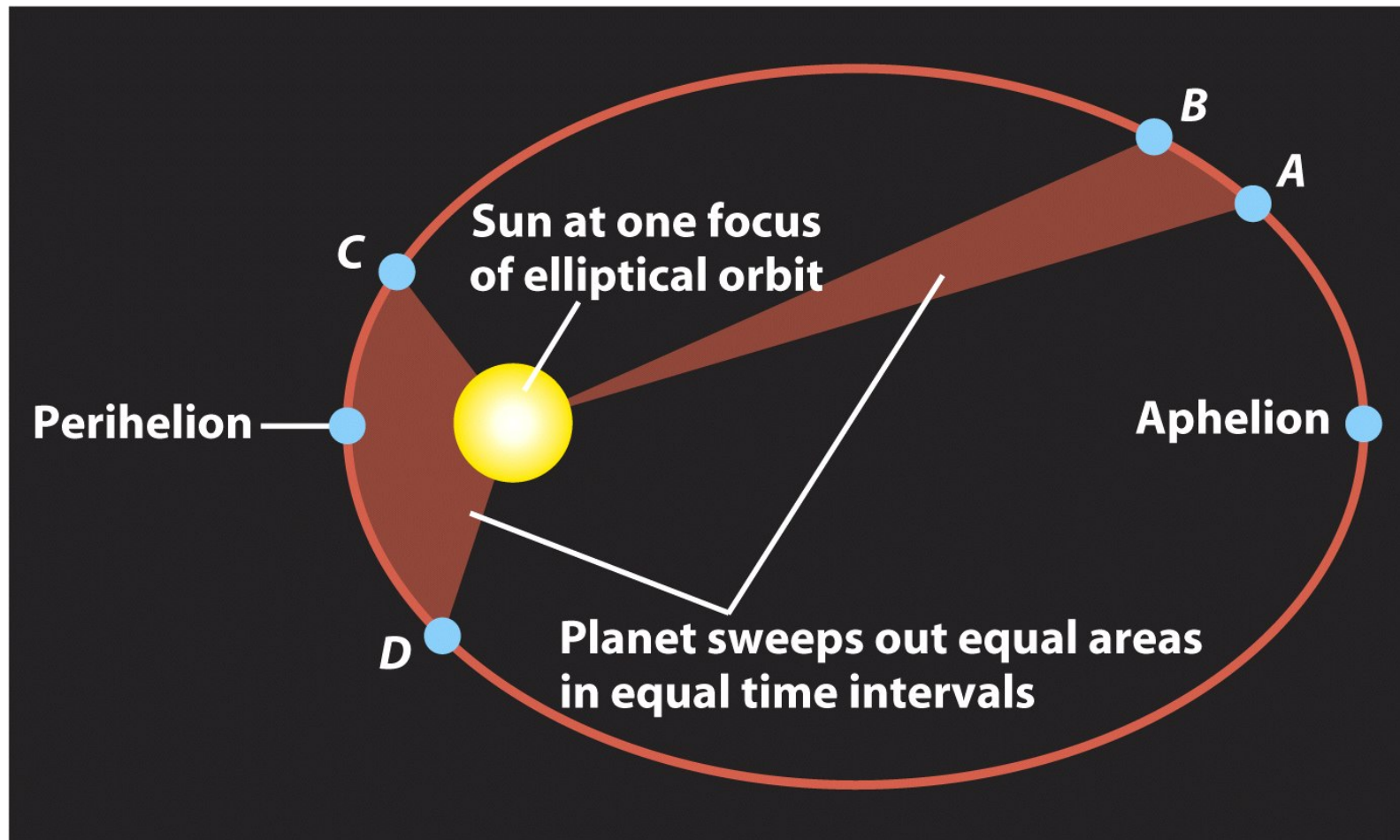
A planet is at the **Aphelion** where it is farthest from the Sun, and at **Perihelion** when it is closest.

$$R_P = b, R_A = a$$



This diagram is WRONG.

Kepler's Second Law: a line joining a planet and the Sun sweeps out equal areas in equal intervals of time.



The speed of the orbital motion varies. It is smaller at Aphelion than at Perihelion.

Ex 1: longitudinal libration of the Moon.

Kepler's Third Law: the square of the sidereal period of a planet is directly proportional to the cube of the semimajor axis of the orbit.

A Demonstration of Kepler's Third Law ($P^2 = a^3$)				
Planet	Sidereal period P (years)	Semimajor axis a (AU)	P^2	a^3
Mercury	0.24	0.39	0.06	0.06
Venus	0.61	0.72	0.37	0.37
Earth	1.00	1.00	1.00	1.00
Mars	1.88	1.52	3.53	3.51
Jupiter	11.86	5.20	140.7	140.6
Saturn	29.46	9.55	867.9	871.0
Uranus	84.10	19.19	7,072	7,067
Neptune	164.86	30.07	27,180	27,190
Pluto	248.60	39.54	61,800	61,820

$$P^2 = a^3$$

P = planet's sidereal period, in years

a = planet's semimajor axis, in AU

Ex 2: A comet has a very long thin elliptical orbit. It is 2 AU from the Sun at the aphelion. What is its sidereal period?

- (a) 2.8 years
- (b) A bit shorter than 1 year
- (c) A bit longer than 1 year
- (d) cannot tell

Ex 3: Comet Halley orbits the Sun with a sidereal period of 76 years.

(a) Find the semi-major axis of its orbit.

(b) At aphelion, the comet is 35.4 AU away from the Sun. How far is it from the Sun at perihelion?

(c) what's the eccentricity of its orbit?

$$(a) P^2 = 76^2 = 5776 = a^3, a = 5776^{1/3} = 18 \text{ (AU)}$$

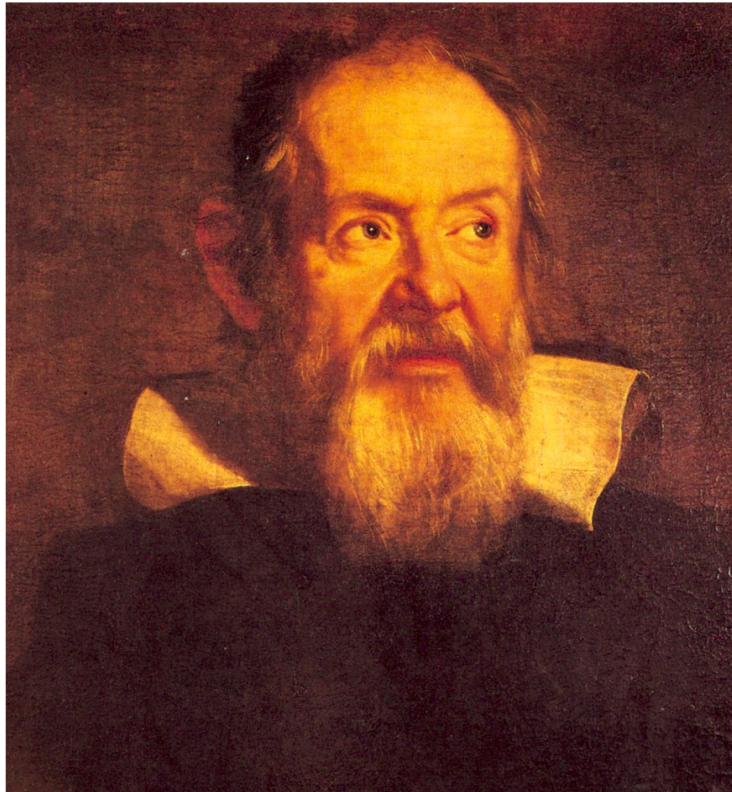
$$(b) R_A + R_P = 2a, R_P = 2a - R_A = 0.6 \text{ (AU)}$$

$$(c) e = (R_A - R_P) / (R_A + R_P) = (35.4 - 0.6) / 36 = 0.915$$

Comet Halley's orbit is very eccentric: the comet is 0.6 AU from the Sun at perihelion and 35.4 AU at aphelion.

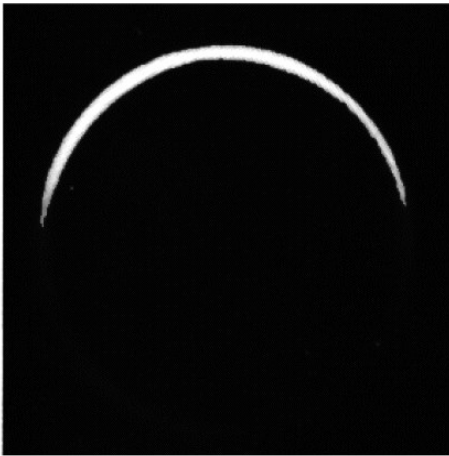
13.3 Galileo's Discoveries

Galileo's discoveries with a telescope strongly supported a heliocentric model.

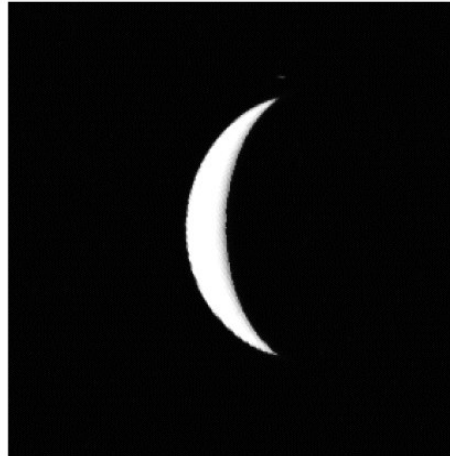


- The invention of the telescope led Galileo to new discoveries that supported a heliocentric model.
- These included his observations of the phases of Venus and of the motions of four moons around Jupiter.

Galileo's telescopic observations constituted the first fundamentally new astronomical data in almost 2000 years. His discoveries strongly suggested a heliocentric structure of the universe.

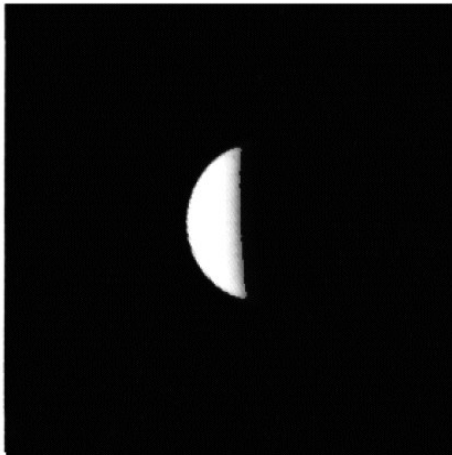


$\alpha = 58^\circ$

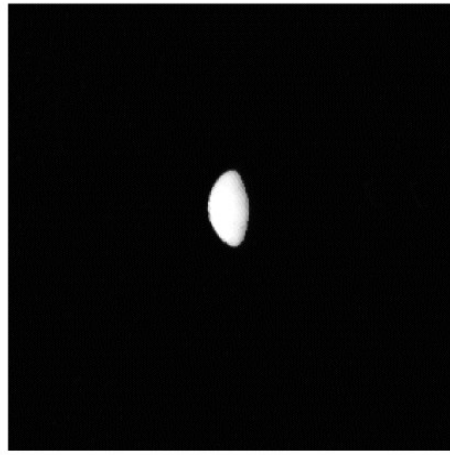


$\alpha = 42^\circ$

The phases of Venus:
one of Galileo's most
important discoveries
with his telescope.



$\alpha = 24^\circ$

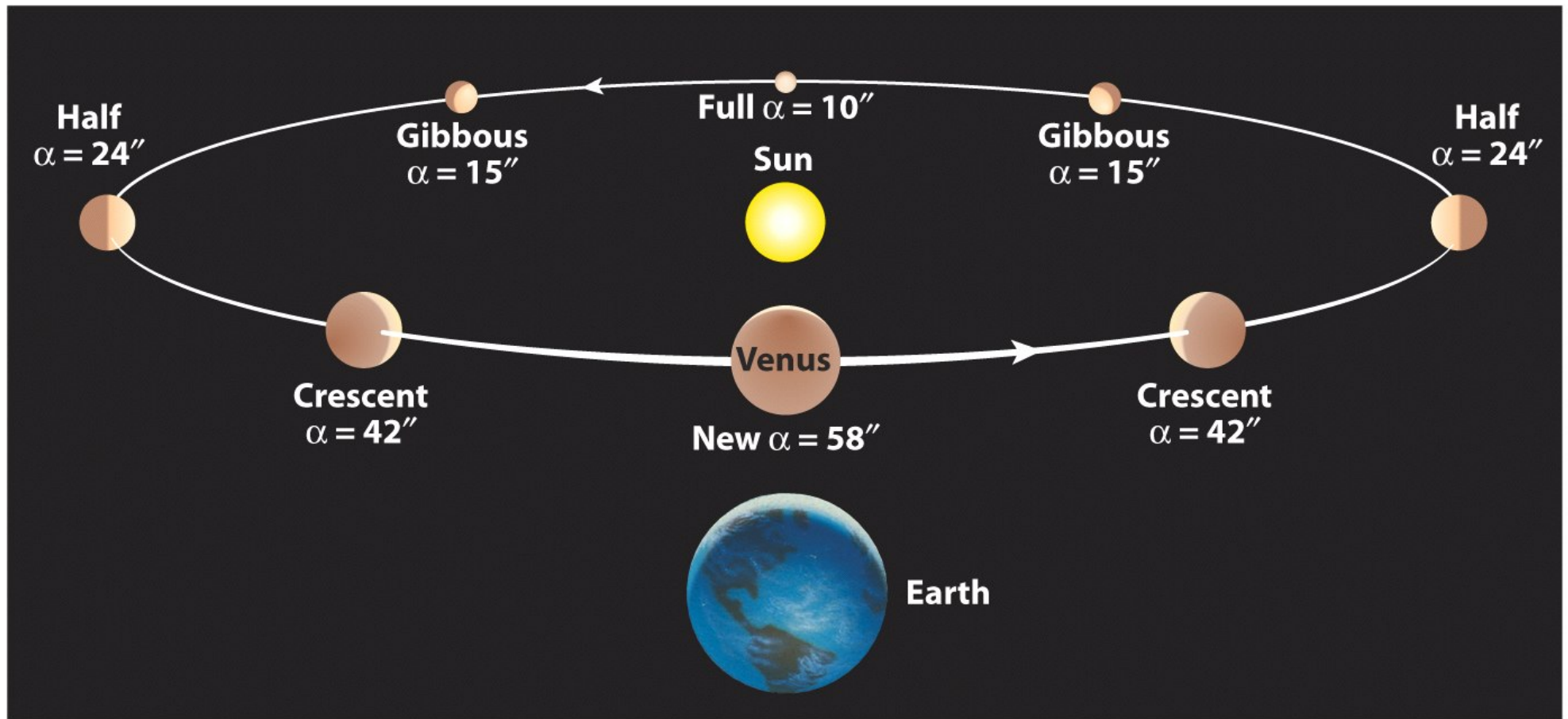


$\alpha = 15^\circ$



$\alpha = 10^\circ$

- Venus exhibits phases like those of the Moon.
- The apparent size of Venus is related to the planet's phase.
- Venus looks small at gibbous phase and large at crescent phase



A heliocentric model, in which both the Earth and Venus orbit the Sun, provides a natural explanation for the changing appearance of Venus.

Jupiter and its Largest Moons



The orbital motions of Jupiter's moons around Jupiter resemble the motions of the planets around the Sun.

13.4 Newton's Law of Universal Gravitation

Newton's Law of Universal Gravitation accounts for Kepler's laws and explains the motions of the planets.

$$F_G = G \frac{mM}{r^2}$$



F = gravitational force between two objects (in newtons)

m = mass of the planet (in kilograms)

M = mass of the Sun (in kilograms)

r = distance between planet and Sun (in meters)

$G = 6.67 \times 10^{-11}$ newton \cdot m²/kg² ,
universal constant of gravitation

$$F_G \sim M, m$$

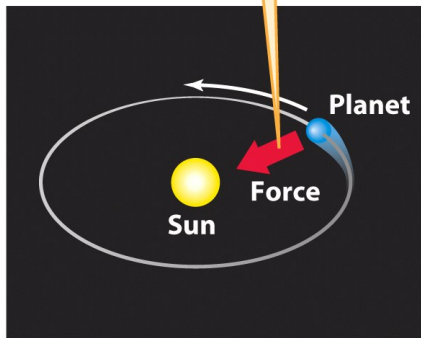
$$F_G \sim 1/r^2$$

Newton's form of Kepler's third law

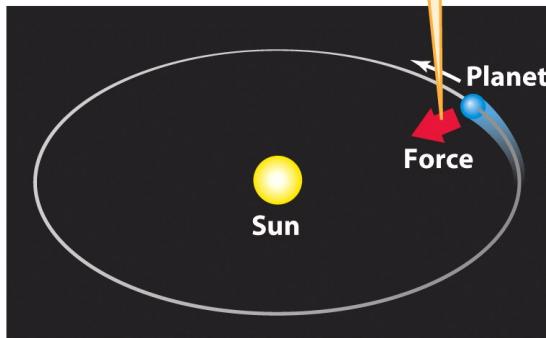
The gravitational (pull or attractive) force keeps the planets orbiting the Sun and satellites orbiting the planets.

$$F = m\omega^2 r = \frac{4\pi^2 mr}{P^2} = F_G$$

To make a planet move at a high speed in a small orbit requires a strong gravitational force.



To make the same planet move at a low speed in a larger orbit requires only a weak gravitational force.



Newton demonstrated that Kepler's third law follows logically the law of gravity, and can be re-written as:

$$a^3 = \left[\frac{GM}{4\pi^2} \right] P^2$$

P = sidereal period, in seconds

a = semimajor axis, in meters

M = mass of the Sun, in kg

G = universal constant of gravitation
= 6.67×10^{-11}

Scaling the equation to Sun-Earth system: Kepler's third law.

Ex 4: Sun's gravitational force to Earth is F_1 . The gravitational force of Earth to Sun is F_2 . F_1 is 300,000 times F_2 , since the mass of Sun is 300,000 times the mass of Earth. True or false?

However, acceleration of an object by a force is inversely proportional to its mass; therefore, subject to the **same** amount of gravitational force, the Sun (star) only wobbles a little bit.

An object in an orbital motion, such as the planet revolving about the Sun, is accelerated by the gravitational force. An acceleration changes the **magnitude** and/or **direction** of a velocity.

Gravitational force by the Sun provides the centripetal force that maintains the orbital motion of a planet around the Sun.

$$a^3 = \left[\frac{GM}{4\pi^2} \right] P^2$$

Ex 5: (assume the orbit is circular) If the mass of the Sun were greater by a factor of 4, while a planet's orbit size remained the same, would the planet revolve faster or slower? By how much?

Ex 6: (assume the orbit is circular) If the mass of the Sun were greater by a factor of 4, and a planet's sidereal period remained the same, would the planet be closer to the Sun or farther away from the Sun? By how much?

Would the planet move faster on its orbit?

Scaling Newton's law of planet motion to the Sun-Earth system, we obtain Kepler's third law.

$$a_{Planet}^3 = \left[\frac{GM_{Sun}}{4\pi^2} \right] P_{Planet}^2$$

$$a_{Earth}^3 = \left[\frac{GM_{Sun}}{4\pi^2} \right] P_{Earth}^2$$



$$\left(\frac{a_{Planet}}{a_{Earth}} \right)^3 = \left(\frac{P_{Planet}}{P_{Earth}} \right)^2$$

This same scaling relation applies to asteroids and comets that revolve around the Sun.

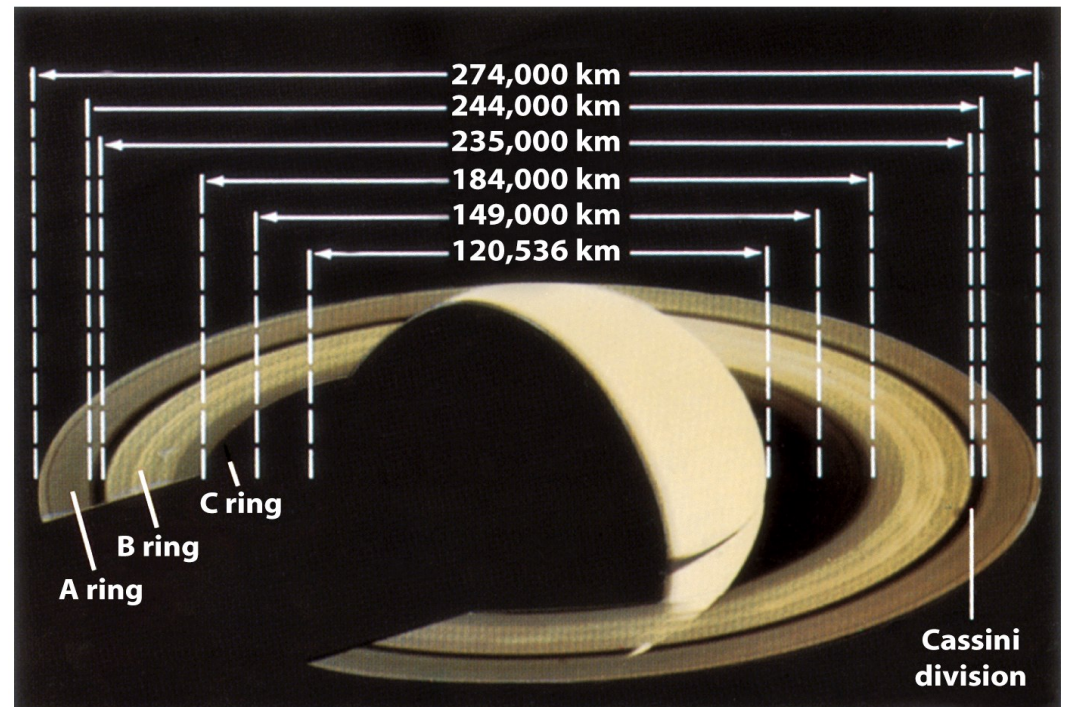
Newton's form of Kepler's third law applies to all situations where two objects orbit each other solely under the influence of their mutual gravitational attraction.

Ex 7: Europa's orbital period around Jupiter is twice that of Io. Calculate the distance of Europa from Jupiter relative to Io's distance to Jupiter.

$$\begin{aligned}a_{Io}^3 &= \left[\frac{GM_{Jupiter}}{4\pi^2} \right] P_{Io}^2 \\a_{Europa}^3 &= \left[\frac{GM_{Jupiter}}{4\pi^2} \right] P_{Europa}^2 \\&= \left[\frac{GM_{Jupiter}}{4\pi^2} \right] (2P_{Io})^2 = 4a_{Io}^3 \\a_{Europa} &= 1.6a_{Io}\end{aligned}$$

Ex 8: The distance of the outer edge of Saturn's A ring is about twice the distance of the inner edge of Saturn's C ring.

- a. A ring particle (A) at the outer edge of A ring revolves faster than a ring particle (C) at the inner edge of C ring by a factor of about 3.
- b. (A) revolves slower than (C) by a factor of 3.
- c. (A) revolves faster by a factor of 2.
- d. (A) revolves slower by a factor of 2.



Ex 9 [previous year's final exam]: Suppose that we discover another moon called Koon orbiting the Earth in an elliptical orbit with a period of 8 months. It is further found that the apparent size of Koon at perihelion is 3 times its apparent size at aphelion. Find the semi-major axis, aphelion distance, and perihelion distance of Koon's orbit **as how many times** the Earth-Moon distance. [HW A1]

Hint: (a) Kepler's third law but scaled to Earth-Moon system;
(b) Kepler's first law.

$$\begin{aligned}
 a_{Moon}^3 &= \left[\frac{GM_{Earth}}{4\pi^2} \right] P_{Moon}^2 \\
 a_{Koon}^3 &= \left[\frac{GM_{Earth}}{4\pi^2} \right] P_{Koon}^2
 \end{aligned}
 \quad \Rightarrow \quad
 \left(\frac{a_{Koon}}{a_{Moon}} \right)^3 = \left(\frac{P_{Koon}}{P_{Moon}} \right)^2$$

Newton's form of Kepler's third law may be used to find the mass of celestial objects in a binary system.

Ex 10: astronomers discovered an exoplanet orbiting a star. They measured the distance of the planet to the star to be 5 AU and the orbit period of the planet to be 2 earth years. From these measurements, can you tell what is the mass of the star in units of solar mass?

using Newton's form of Kepler's third law for the exoplanet : $a_p^3 = \left[\frac{GM_{star}}{4\pi^2} \right] P_p^2$

And for Earth : $a_E^3 = \left[\frac{GM_{sun}}{4\pi^2} \right] P_E^2$

scale the two equations, we get :

$$\left(\frac{a_p}{a_E} \right)^3 = \left(\frac{M_{star}}{M_{sun}} \right) \left(\frac{P_p}{P_E} \right)^2, \quad \text{so : } \left(\frac{M_{star}}{M_{sun}} \right) = \frac{\left(\frac{a_p}{a_E} \right)^3}{\left(\frac{P_p}{P_E} \right)^2} = 31$$

(optional) Ex 11: precisely, in a binary system, both objects orbit about their center of mass with the same angular speed – star wobbles due to planet perturbation! The exact Newton's form of Kepler's laws yields the total mass of the binary system.

$$\text{Exact Newton's form of Kepler's third law : } a^3 = \left[\frac{G(m_1 + m_2)}{4\pi^2} \right] P^2$$

P : orbital period of the binary

a : semi - major axis of the ellipse made by the separation of the objects

Motion of each of the binary members is given by the ellipse of the same eccentricity, but scaled down according to their mass ratio.

$$P_1 = P_2 , \quad a_1 = \frac{m_2}{m_1 + m_2} a, \quad a_2 = \frac{m_1}{m_1 + m_2} a , \quad \frac{a_1}{a_2} = \frac{m_2}{m_1} , \quad \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

13.5 Gravity in the Solar System

Solar/stellar system is formed by gravitational contraction.

gravitational force between two objects: $F_G = G \frac{mM}{d^2}$

gravitational energy in a binary system: $U = - G \frac{mM}{d}$

gravitational energy of a mass M and radius R : $U = -\frac{3}{5} \frac{M^2 G}{R}$

With decreasing R , U is converted to internal heat.

Kelvin-Helmholtz contraction provides energy for proto-stars before fusion ignition, and is still converting gravitational energy to internal heat in Jupiter.

Particles with low speed are retained by gravity.

$$\frac{1}{2}mv^2 \leq G \frac{Mm}{(R+h)^2}$$
$$v_{esc} = \sqrt{\frac{2GM}{R+h}} \approx \sqrt{\frac{2GM}{R}}$$

Ex 12: launching satellite on Earth

Ex 13: capture of the atmosphere by a planet
(N.B.: compare $6v_{th}$ and v_{esc})

Tidal force (differential gravitational force) accounts for many phenomena in the solar system.

The tidal force by a planet mass M on the unit mass at two edges of a satellite of radius R and distance d .

$$f_{tidal} = G \frac{4MR}{d^3}$$

Ex 14: examples of tidal force effect in the solar system.

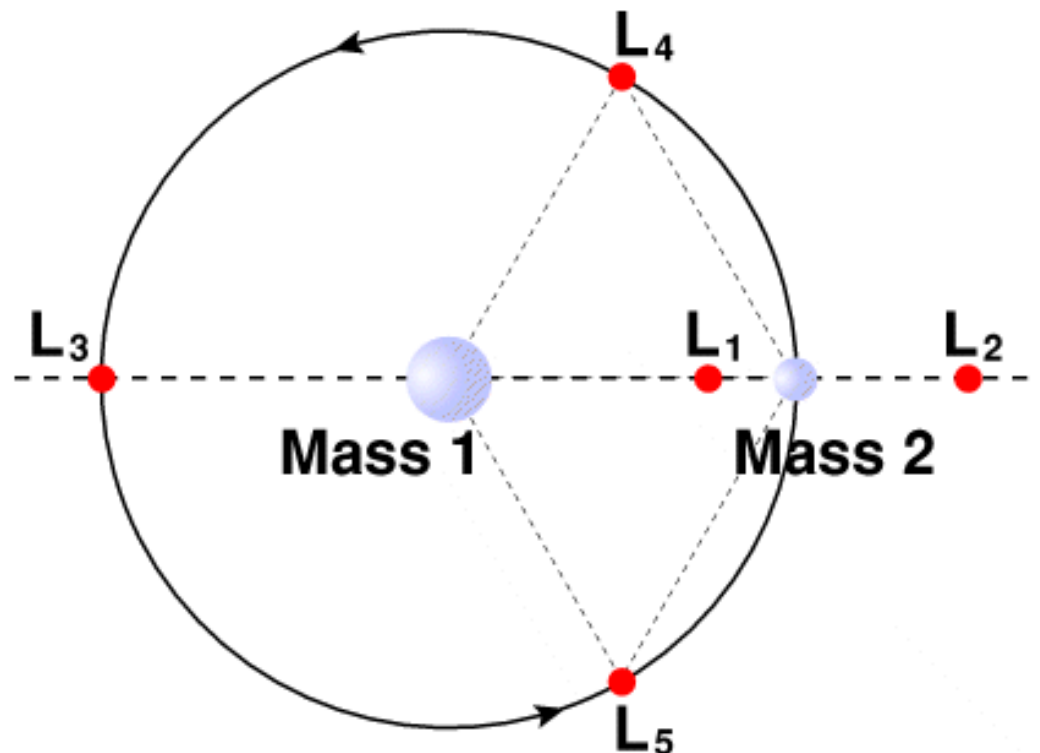
- tides on earth; tidal bulges of planets and moons;
- synchronous rotation of many moons in the solar system
- 3-to-2 spin-to-orbit coupling of Mercury's motion
- future of Moon and Triton
- tidal heating of Jovian moons
- Roche limit and Jovian rings

Gravity shapes the solar system in many ways.

Ex 15: shepherd satellites and gaps in Saturn's rings.

Ex 16: formation and configuration of asteroid belt.

Ex 17: At Lagrange points L_1 and L_2 , an object and Mass 2 orbit Mass 1 with the same rate. How does that happen?



Key Words

- ellipse
- gravitational force
- heliocentric model
- Kepler's laws
- law of equal areas
- law of universal gravitation
- major axis
- Newton's form of Kepler's third law
- semimajor axis
- sidereal period
- synodic period
- universal constant of gravitation