

Assignment 1: due Aug 27

Problems to be submitted:

1. The gravitational force on an object of mass m at the height h above the surface of the earth (mass M and radius R) is precisely given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r},$$

where $r = R + h$, and $\hat{r}$ is the unit vector indicating the radial direction.

- Re-write the expression of $\vec{F}$ as a function of $x \equiv \frac{h}{R}$, with a constant multiplier. State the units of that constant in terms of base units kg, m, s.
- Expand the expression of F in the power series of x .
- For $h < R$, keep the first non-zero term of the power series, and show that the gravitational force with this approximation becomes a constant $F = mg$, and find the expression of g .
- For $h = 0.01R$, evaluate the accuracy of your solution in (c) by finding the ratio of your solution in (c) to the exact solution.

2. In 1900, the physicist Max Planck has found a mathematical description, the Planck function, of electromagnetic radiation by an opaque object in thermal equilibrium (for example, a star), also referred to as blackbody radiation. It resolved the ultraviolet catastrophe in classic physics and led to new theories in physic. The Planck function describes the radiation by the object as a function of frequency ν (in units Hz) of the EM wave, given the object's constant temperature T (in kelvin),

$$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(\frac{h\nu}{kT}) - 1}.$$

Here ν is frequency, h is Planck's constant, c speed of light, and k Boltzmann constant.

- Define a dimensionless variable $x \equiv h\nu/kT$, re-write the Planck function in terms of x , and show that the constant multiplier has the units $\text{J s}^{-1} \text{m}^{-2} \text{Hz}^{-1}$ – this is the units of specific intensity.
- Expand the denominator into a power series of x , and find the approximate solution of the Planck function by keeping only the first term of the series. This function is called the Rayleigh-Jean's law of radiation.
- For $T = 6000\text{K}$, compare the radiation computed by Planck's function with that computed by Rayleigh-Jeans law, at the wavelengths $\lambda = 1\text{m}$ and then $\lambda = 100\text{nm}$, respectively. What is your conclusion? Hint: what is the relation between wavelength and frequency? In each case, what is the value of x ?
- Prior to the introduction of Planck's function, physicists used Rayleigh-Jean's law to describe EM radiation and became worried that the radiation at short-wavelength (such as the ultraviolet light) would be unreasonably high. This is known as the ultraviolet catastrophe. Can you describe how Planck resolved the catastrophe?

3. (a) Expand the function $\sin(x)$ into the power series around $\frac{\pi}{2}$.

(b) For $x = 1.01\frac{\pi}{2}$, at which term, the difference between your approximate solution in (a) and the

exact solution is less than 0.1% of the exact solution? You may use a calculator to find the exact solution.

(c) Now if we use the Maclaurin series (Boas 13.1), at which term, the difference between the approximate solution and the exact solution is less than 0.1% of the exact solution?

4. Boas Chap 1, Section 15, Problem 30.

Solutions:

1. [2+4+2+2] (a) We can rewrite the expression of the force as

$$\vec{F} = -\frac{GMm}{R^2} \frac{1}{(1+x)^2} \hat{r},$$

with $x \equiv h/R$ being **dimensionless**. The constant multiplier is

$$-\frac{GMm}{R^2},$$

which has the dimension of force, in units of Newton, or kg m s^{-2} .

(b) This is a binomial expansion of the function $f(x) = (1+x)^{-2}$, which can be expanded as

$$f(x) \equiv (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

So the expression of the force is given by

$$F = -\frac{GMm}{R^2} (1 - 2x + 3x^2 - 4x^3 + \dots).$$

(c) For $x \ll 1$, the series in (b) converges, and the first non-zero term of the power series is a constant,

$$F_0 \approx -\frac{GMm}{R^2} = mg,$$

where

$$g \equiv -\frac{GM}{R^2},$$

is the constant gravitational acceleration at a height much smaller than the radius of Earth.

(d) For $x = 0.01$ (which is 60 km), we can find the ratio of the solution in (c) to the exact solution

$$\frac{F_0}{F} = (1 + 0.01)^2 \approx 1.02,$$

so the difference between the exact solution and approximate solution is about 2%, at the height of 60 km!

(2) [2+3+4+1] (a) We re-write the Planck function in terms of $x = h\nu/kT$ by

$$B(x) = \frac{2k^3T^3}{h^2c^2} \frac{x^3}{e^x - 1}.$$

Here kT has units of energy J, $h\nu$ also has the units of energy J, so x is **dimensionless**, and the second term as a function of x is also dimensionless. The constant multiplier has the units of $\text{J}^3 \cdot \text{J}^{-2} \cdot \text{H}_z^2 \cdot \text{m}^{-2} \cdot \text{s}^2$, which is same as $\text{J m}^{-2} \text{s}^{-1} \text{H}_z^{-1}$, the units of specific intensity of light.

(b) Here the expression dependent on x is

$$f(x) = \frac{x^3}{e^x - 1},$$

and the denominator can be expanded as

$$e^x - 1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

If we only keep the first non-zero term of the denominator, the Planck function becomes

$$B_0(x) \approx \frac{2k^3 T^3}{h^2 c^2} x^2.$$

(c) The relationship between wavelength λ and frequency ν is $\nu = c/\lambda$. For $\lambda = 1\text{m}$, we find $x = 2.40 \times 10^{-6}$, and the ratio of the approximate solution to the exact solution is

$$\frac{B_0(x)}{B(x)} = \frac{e^x - 1}{x} \approx 1 + \frac{x}{2} = 1 + 1.20 \times 10^{-6}.$$

You can see that the two solutions from Planck's function or Rayleigh-Jean's law are almost identical. Note that you might not be able to get a correct answer if you directly use your calculator to compute the ratio, since the ratio is very close to evaluating zero over zero! Now with $\lambda = 100\text{nm}$, $x = 24.02$, the ratio now becomes

$$\frac{B_0(x)}{B(x)} = \frac{e^x - 1}{x} \geq 1.13 \times 10^9.$$

So, in the long-wavelength limit, the Planck function and Rayleigh-Jean's law give the same solution, but in the short-wavelength limit, the Rayleigh-Jean's law produces a very large radiation compared with the Planck function!

(d) The very large short-wavelength radiation predicted with Rayleigh-Jean's law, which is valid for long-wavelength radiation, is called the ultraviolet catastrophe. Planck introduced the correct expression for radiation in all wavelengths, which can reconcile with Rayleigh-Jean's law in the long-wavelength range, but has also resolved the ultraviolet catastrophe. As λ decreases, or ν increases, Rayleigh-Jean's law is not a good approximation at all, but with the Planck function, the exponential term in the denominator grows fast, and short-wavelength radiation does not blow up.

3. [4+2+2] (a) We use Taylor expansion $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \dots + \frac{1}{n!}f^n(x_0)(x - x_0)^n + \dots$ to find

$$\begin{aligned} \sin(x) &= \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{6}\cos\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^3 + \frac{1}{24}\sin\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^4 + \dots \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 + \dots \end{aligned}$$

(b) At $x = 1.01\frac{\pi}{2}$, $\sin(x) = 0.999877$ – what value do you get with your calculator?! If we are not too sure about the limit of a calculator, we can explore that with the series expansion! The argument in the expansion is $x - x_0 = 0.01\frac{\pi}{2} = 1.57 \times 10^{-2}$; taking this back to (a) we find

$$\sin(x) \approx 1 - \frac{1}{2} \times \left(0.01\frac{\pi}{2}\right)^2 + \frac{1}{24} \times \left(0.01\frac{\pi}{2}\right)^4 \dots = 1. - 1.23 \times 10^{-4} + 2.53 \times 10^{-9} \dots$$

So at the second term, the difference is smaller than 0.1% of first term, regardless of the accuracy of the “exact” solution computed by a calculator, since the contribution by the third term and beyond would be smaller than the first term by at least 8 orders of magnitude!

(c) With the Maclaurin series (Boas 13.1), the expansion is, instead,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

It can be found that, the solution oscillates with a larger amplitude than in the case of (a); the contribution of the fourth term is of order 5.02×10^{-3} , and that of the fifth term is of order 1.75×10^{-4} . So at the fifth term, the difference between the approximate solution and the exact solution (which is nearly 1.) is less than 0.1%. From this exercise, we arrive at the conclusion: to approach the solution quicker, we want to expand the function as a power series of the argument $y = x - x_0 \ll 1$ that converges faster than that of x !

4. [2+2] (a) Define the half distance between the car and the tree as L , we can write

$$\sin \theta = \frac{x}{\sqrt{x^2 + L^2}} = \frac{x}{L} \left(1 + \frac{x^2}{L^2}\right)^{-\frac{1}{2}}.$$

So the tension force is given by

$$T = \frac{FL}{2x} \left(1 + \frac{x^2}{L^2}\right)^{\frac{1}{2}}.$$

The expansion with respect to x is a binomial expansion, therefore,

$$T = \frac{FL}{2} x^{-1} \left(1 + \frac{1}{2} \frac{x^2}{L^2} - \frac{1}{8} \frac{x^4}{L^4} + \dots\right).$$

(b) In this case, we can expand $\sin \theta$ to the power series of θ

$$\sin \theta = \theta \left(1 - \frac{1}{6} \theta^2 + \frac{1}{120} \theta^4 \dots\right).$$

And the expansion of $1/\sin \theta$ also becomes a binomial expansion

$$\frac{1}{\sin \theta} = \theta^{-1} \left(1 - \frac{1}{6} \theta^2 + \frac{1}{120} \theta^4 - \frac{1}{5040} \theta^6 + \dots\right)^{-1} = \theta^{-1} \left(1 + \frac{1}{6} \theta^2 + \frac{7}{360} \theta^4 + \dots\right).$$

Here we take

$$y = -\frac{1}{6} \theta^2 + \frac{1}{120} \theta^4 - \frac{1}{5040} \theta^6 + \dots,$$

and expand the expression into the power series of y , and then replace y with the powers of x , and re-organize terms to get the above solution. So the tension force is expanded as

$$T = \frac{F}{2 \sin \theta} = \frac{F}{2} \theta^{-1} \left(1 + \frac{1}{6} \theta^2 + \frac{7}{360} \theta^4 + \dots \right).$$

The series is different depending on the argument of the expansion. Both series converge at any value of the argument. You can prove that, given enough number of terms, the two series would lead to the same solution, but one series may converge faster than the other depending on the value of the argument.